RESEARCH STATEMENT: BOULANGER JULIEN

1. Introduction

My research focuses on several geometric and dynamical questions related to translation surfaces. Translation surfaces lie at the crossroad between (Riemannian and algebraic) geometry, low-dimensional dynamics, ergodic theory, geometric group theory, Teichmüller theory, and number theory. In the first part of my thesis, I was specifically interested in the algebraic intersection of closed curves on these surfaces, and more precisely the question of knowing *how many times two closed curves of a given length can intersect.* The second part of my thesis is devoted to the study of *Veech groups* of translation surfaces, which somehow encode the symmetries of each given surface and which are Fuchsian groups. A fundamental (but difficult) question is to know *which Fuchsian groups can be realized as Veech groups of translation surfaces.*

Informally, a translation surface can be seen as a collection of polygons in the plane with parallel sides identified by translations. Such a surface inherits a flat metric except on a finite number of conical singularities whose angles are integer multiples of 2π, and which correspond to certain vertices of the polygons. The set of all translation surfaces[1] (of genus $g \geq 1$) has a natural geometric structure and can be interpreted as the Hodge bundle $\Omega\mathcal{M}_g$ of the holomorphic 1-forms above the moduli space $\mathcal{M}_g$ of complex structures on a (compact orientable) genus g surface. It is naturally stratified by the number and the order of the singularities; namely $\Omega\mathcal{M}_g = \sqcup\mathcal{H}_g(k_1, \dots, k_n)$, where the k_i are integers representing the order of the singularities satisfying $\sum_{i=1}^{n} k_i = 2g - 2$.

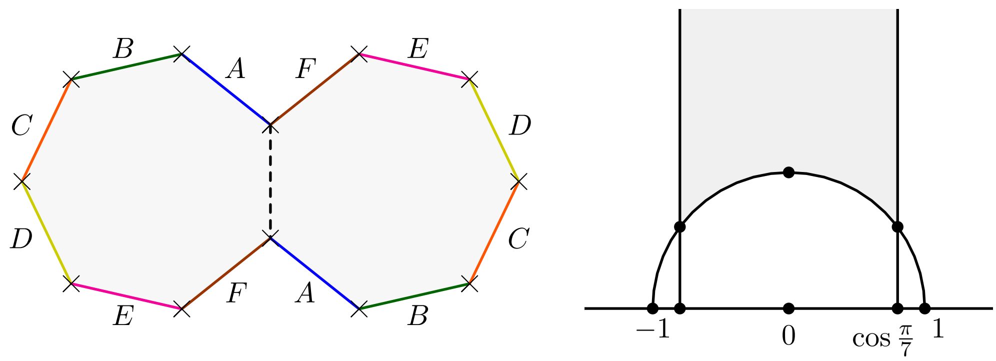

Figure 1. On the left, the double heptagon translation surface. It has genus three and a single singularity of angle 10π. On the right, a fundamental domain in the Poincaré half-plane $\mathbb{H}^2$ for the orbit of the double heptagon under the action of $SL_2(\mathbb{R})$. Each point corresponds to an element of the orbit and the double heptagon lies at the two identified corners. The Veech group of the double heptagon is conjugated to the Hecke group of level seven.

This structure proves to be particularly rich, and in particular it carries a natural action of $GL_2^+(\mathbb{R})$ which comes from the action on the polygons. As shown by Veech, the stabilizer of a given $X \in \Omega\mathcal{M}_g$ is a Fuchsian group, which is now referred to as the Veech group of X. Every element of the Veech group induces an[2] *affine diffeomorphism* of the surface. The work of several authors, including Veech, has shown that this group encodes many geometric properties of the surface. The translation surfaces whose Veech group is a lattice (called Veech surfaces)

[1] modulo the action of the mapping class group.

[2] In some cases, it may also induce several different affine diffeomorphisms.

form a class of particularly interesting surfaces, both from the geometrical and dynamical point of view. The orbit of these surfaces under the action of $SL_2(\mathbb{R}) \subset GL_2^+(\mathbb{R})$ projects in $\mathcal{M}_g$ to a complex geodesic, which is called a *Teichmüller curve.* The classification of Teichmüller curves, and more generally of orbit closures under the $SL_2(\mathbb{R})$-action, is a central problem in Teichmüller dynamics where recent progress lead to the developpement of many new and revolutionnary techniques.

Organization. In the sequel, I will describe the following.

- In Section 2, I adress the question of the classification of Teichmüller curves by explaining a result joint with Sam Freedman [BF22] which shows that there is no geometrically primitive Veech surface in the locus $Prym(2,2) \subset \mathcal{H}(2,2)$ of surfaces having a *Prym involution.*
- In Section 3, I focus on translation surfaces with infinitely generated Veech groups, and more specifically on the study of *connection points* that allow to build such surfaces. In [Bou22], I prove that the central points of the double heptagon are not connection points, which gives a negative answer to a question of P. Hubert and T. Schmidt. This question is related to the distribution of *saddle connections* as well as the notion of *Hecke continued fractions*, and exhibits a deep link with a problem from number theory.
- In Section 4, I elaborate upon the notion of *interaction strength* which quantifies the maximum number of intersection of two curves of given length on a Riemannian surface with conical singularities. I will specifically address the case of flat surfaces and the behavior of this flat interaction strength by deformation of a given surface, which I studied in [BLM22], [Bou23a] and [Bou23b] as well as in an ongoing work with I. Pasquinelli.

2. Veech surfaces with a Prym involution

One method for classifying Veech surfaces may be to first search for those that cannot be constructed from another Veech surface by a covering construction. Such surfaces are called *geometrically primitive.* Despite several recent major advances such as the results of [EFW18] and [EMMW20], this question of the classification of primitive Veech surfaces remains largely open. In genus two, primitive Veech surfaces have been classified in a series of articles by C. McMullen [McM05b, McM05a, McM06a]. In genus three, the classification is not complete but C. McMullen [McM06b] exhibited an infinite family of primitive Veech surfaces. The work of several authors has since shown that, apart from this family, there is only a finite number of primitive Veech surfaces (up to the action of $GL_2^+(\mathbb{R})$), see [McM21, Theorem 5.5]. This infinite family consists of surfaces having a *Prym involution* and a single singularity. A translation surface with a Prym involution can be constructed as a double cover of a *half-translation surface* (branched at the singularities), see [McM06b] or [LN13]. To search for other primitive Veech surfaces, a natural idea is to look for translation surfaces with a Prym involution, but having several singularities. In this context, E. Lanneau and M. Möller [LM19] searched for primitive Veech surfaces in the loci $Prym(2,1,1)$ and $Prym(2,2)$, which is made up of translation surfaces having a Prym involution and respective singularities of order $(2,1,1)$ and $(2,2)$. They prove that there is no primitive Veech surface in $Prym(2,1,1)$ and identify 92 candidates in $Prym(2,2)$. In a joint work with Sam Freedman [BF22] published at the *Comptes-Rendus de l'académie des sciences*, we show:

Theorem 2.1. *There are no geometrically primitive Veech surfaces in* $Prym(2,2)$.

Our method relies on a computer program which uses the package *Flatsurf* of *Sage* and where we compute the orbit closure of the 92 candidates. It would be interesting to know whether the methods of [LM19] and [BF22] could be extended to $Prym(1,1,1,1)$.

3. Connection points and Hecke continued fractions

In another direction, we can look for translation surfaces whose Veech group is infinitely generated. A way of building such surfaces has been exhibited by P. Hubert and T. Schmidt

[HS04], who define and use the notion of *connection point*. A connection point is a (non-singular) point of the surface such that any geodesic starting at a singularity and passing through this point again meets a singularity. A vertex-to-vertex geodesic trajectory is a *saddle connection*. A (non-singular) point of the surface is said to be *periodic* if its orbit under the action of the affine diffeomorphisms of the surface is finite. A periodic point is automatically a connection point. In this context, we can ask:

Question. *Is it possible to characterize the connnection points of a given translation surface?*

By a result of C. McMullen (see also Boshernitzan [Bos88] in the setting of interval exchange transformations), when the field generated by the traces of the matrices of the Veech group (called the *trace field*), is either $\mathbb{Q}$ or quadratic over $\mathbb{Q}$, the connection points are exactly (after a natural normalization) the points with coordinates in the trace field. However, as soon as the degree over $\mathbb{Q}$ of the trace field is three or more, no example of a non-periodic connection point is known. In [Bou22], published at the *Bulletin de la SMF*, we are specifically interested in the double heptagon, built from two copies of a regular heptagon whose sides are glued together, and more generally the double n-gon for $n \geq 7$ odd. These surfaces are, with the regular n-gons for even n, the original surfaces studied by Veech in his founding article [Vee89]. Their trace fields have degree at least three over $\mathbb{Q}$ for odd $n \geq 7$. They are therefore natural examples on which one can start looking for non periodic connection points. In particular, the centers of the n-gons, which are not periodic points, are candidates to be connections points. For $n = 7$ and $n = 9$, we show that in fact this is not the case:

Theorem 3.1. *The central points of the double heptagon (as well as the double nonagon) are not connection points.*

To prove Theorem 3.1, we use the notion of Hecke continued fractions, introduced by D. Rosen in [Ros54]. It follows from a result of T. Schmidt and M. Sheingorn [SS95] (as well as Veech dichotomy [Vee89]) that these continued fractions characterize the periodic[3] directions on the double n-gon: a direction on the double n-gon is periodic if and only if the Hecke continued fraction expansion of its slope (obtained from the "next-integer" continued fraction algorithm) is finite. In [Bou22], we exhibit geodesic trajectories from starting at the singularity and passing through one of the central points whose slope has an infinite periodic Hecke continued fraction expansion. In particular, such geodesic trajectories are not closed. It is interesting to note that such examples give vertex-to-vertex billard trajectories on the $(\frac{\pi}{2}, \frac{\pi}{7}, \frac{3\pi}{14})$-triangle such that all other trajectories in this direction are uniquely ergodic. To go further, it would be interesting to classify the periodic directions in the trace field for the double heptagon, see for example [McM21, Question 3.8]:

Question. Does every $x \in \mathbb{Q}[2\cos\frac{\pi}{7}]$ have either a finite or an eventually periodic Hecke continued fraction expansion?

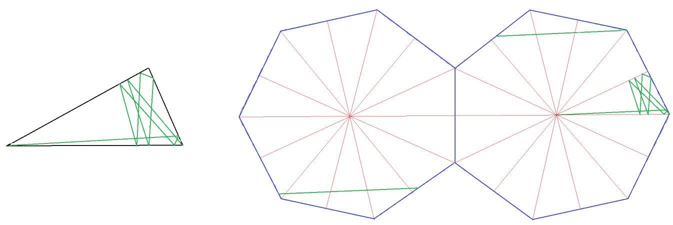

FIGURE 2. A vertex to vertex trajectory on the $(\frac{\pi}{2}, \frac{\pi}{7}, \frac{3\pi}{14})$-triangle in an uniquely ergodic direction.

[3]A direction is periodic if any geodesic in this direction is either closed or a vertex-to-vertex trajectory.

4. Interaction strength on the moduli space of translation surfaces

Finally, a significant part of my thesis deals with the algebraic intersection of curves on a Riemannian surface, defined by assigning a sign to each intersection point, and investigates more specifically the following:

Question. *How many times can two closed curves of a given length intersect?*

This question, which makes sense for any orientable surface X on which we can measure the lengths of the closed curves, can be quantified by defining:

$$\mathrm{KVol}(X) := \mathrm{Vol}(X) \cdot \sup_{\alpha,\beta} \frac{\mathrm{Int}(\alpha,\beta)}{l(\alpha)l(\beta)}$$

where the supremum is taken over pairs of closed curves on X. Normalizing by the volume makes the quantity scalar invariant. According to the terminology of [Tor23], we can refer to KVol(X) as the *interaction strength* of the surface X, but in our setting we consider the algebraic intersection instead of the geometric intersection. The study of KVol goes back to D. Massart's work in [Mas97], where this KVol appears (indirectly) as a comparison constant between the stable norm and the Hodge norm in homology. The study of KVol has since been deepened by D. Massart and B. Muetzel [MM14], but many questions remain open, starting with the question of the explicit computation of KVol on simple examples of surfaces. In this context, S. Cheboui, A. Kessi and D. Massart [CKM21a, CKM21b] initiated the study of KVol on translation surfaces by explicitely calculating KVol on the Teichmüller disk of some square-tiled surfaces. In [BLM22] and [Bou23a], we study KVol on the Teichmüller disk of the original Veech surfaces: the double regular n-gon for odd n and the regular n-gon for odd n.

Namely, we first use a *subdivision method* (we decompose closed geodesics in small pieces for which we can estimate both lengths and intersections) to obtain

Theorem 4.1. *Let $n \geq 2$, then:*

- *The supremum in the definition of KVol on the double $(2n+1)$-gon is achieved uniquely by pairs of distinct sides of the polygons.*
- *The supremum in the definition of KVol on the regular $4n$-gon is achieved uniquely by pairs of distinct sides of the polygons.*

Note that the regular $(4n+2)$-gon is a translation surface with two singularities, and the sides are not closed curves anymore. In this case, our techniques only allow to obtain an upper bound on KVol, see [Bou23a].

Behavior of KVol under deformation. Next, we study the behavior of KVol on these surfaces under deformation. For this, we use the knowledge of KVol on the double regular $(2n+1)$-gon (resp. the $4n$-gon) as well as their associated staircase models, and we prove and use a sharp result of hyperbolic geometry that allows to compare the angles between two pairs of saddle connections on a translation surface. This method applies particularly well to the double regular $(2n+1)$-gon, for which we have:

Theorem 4.2. [BLM22, Theorem 1.1] *Let $n \geq 2$. Given $d, d' \in \mathbb{R} \cup \{\infty\} \simeq \partial\mathbb{H}^2$, let $\gamma_{d,d'}$ the geodesic of the Poincaré half plane $\mathbb{H}^2$ whose endpoints are d and d'. Let $X = M \cdot \mathcal{S}_{2n+1}$ be a surface in the Teichmüller disk of the double regular $(2n+1)$-gon X_{2n+1} which is obtained from the staircase model $\mathcal{S}_{2n+1}$ associated to X_{2n+1} by applying a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$. Then, we have:*

$$KVol(X) = K_{2n+1} \cdot \frac{1}{\cosh(\mathrm{dist}_{\mathbb{H}^2}(\frac{di+b}{ci+a}, \Gamma_{2n+1} \cdot \gamma_{\infty,0}))}$$

where $K_{2n+1} = \frac{n}{2}\cot(\frac{\pi}{n}) \cdot \frac{1}{\sin \pi/n} > 0$, Γ_{2n+1} is the Veech group of X_{2n+1} and $\mathrm{dist}_{\mathbb{H}^2}$ is the hyperbolic distance in the Poincaré half plane.

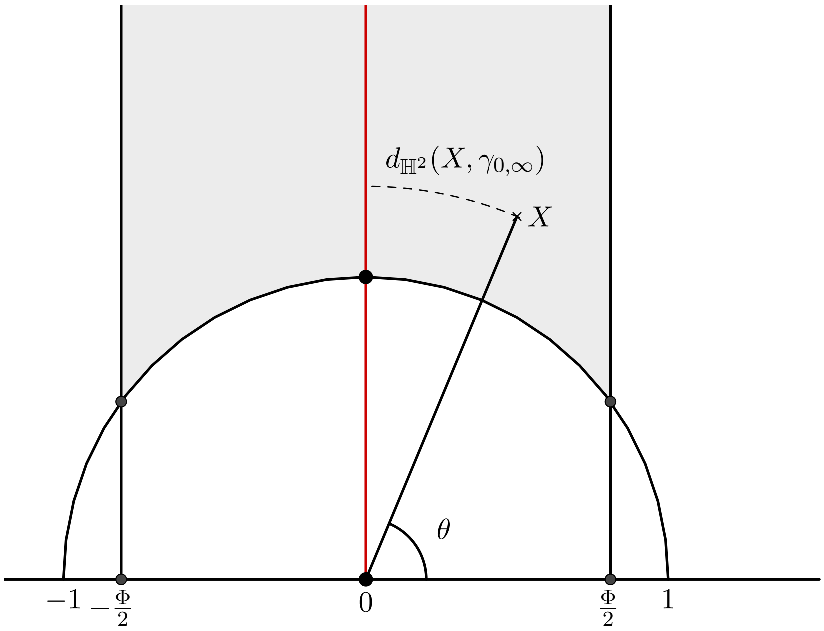

FIGURE 3. In the fundamental domain for the Teichmüller disk of the double pentagon, we have, $\mathrm{KVol}(X) = K_5 \sin\theta = K_5 \frac{1}{\cosh(d_{\mathbb{H}^2}(X,\gamma_{0,\infty}))}$.

A geometric interpretation of this result is given in Figure 3. In a forthcoming paper with I. Pasquinelli (a preliminary version already appears in my thesis manuscript), we generalise this result to Bouw-Möller surfaces with a single singularity.

Concerning the regular $4n$-gon, interestingly, due to the shape of the cylinder decomposition, KVol in the Teichmüller disk does not behave as well as in the double $(2n+1)$-gon case: instead of computing KVol on any point of the fundamental domain by looking at the distance with respect to a single geodesic, there is now an (explicit) infinte family:

Theorem 4.3. [Bou23a, Theorem 1.1] *Let $n \geq 2$. Given $d, d' \in \mathbb{R} \cup \{\infty\} \simeq \partial\mathbb{H}^2$, let $\gamma_{d,d'}$ denote the geodesic in the hyperbolic plane $\mathbb{H}^2$ having d and d' as endpoints, and define:*

$$\mathcal{G}_{max} = \bigcup_{k \in \mathbb{N}^* \cup \{\infty\}} \gamma_{\infty, \pm\frac{1}{k\Phi}}$$

(with the convention $\frac{1}{\infty} = 0$).

Let $X = M \cdot \mathcal{S}_{4n}$ be a surface in the Teichmüller disk of the regular $4n$-gon X_{4n}, obtained from the staircase model of the regular $4n$-gon $\mathcal{S}_{4n}$ by applying a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$. Then, we have:

$$\mathrm{KVol}(X) = K_{4n} \cdot \frac{1}{\cosh(\mathrm{dist}_{\mathbb{H}^2}(\frac{di+b}{ci+a}, \Gamma_{4n} \cdot \mathcal{G}_{max}))} \tag{4.1}$$

Where $K_{4n} > 0$ is an explicit constrant which only depends on n and $\mathrm{dist}_{\mathbb{H}^2}$ denotes the hyperbolic distance.

Lower bound on KVol on the minimal stratum. Finally, in [Bou23b] we study KVol on the minimal stratum of the moduli space of translation surfaces, and namely we are interested in the following:

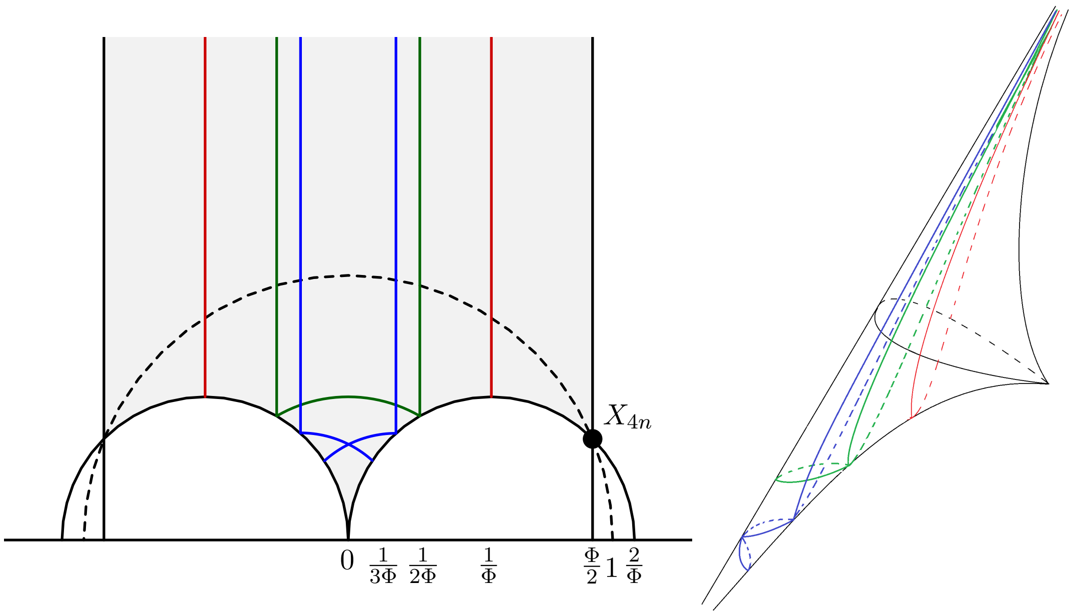

Figure 4. The geodesics $\gamma_{\infty,\frac{1}{k\Phi}}$ for $k = 1, 2, 3$ and their images by the Veech group intersecting the fundamental domain. On the right, the same geodesics on the surface $\mathbb{H}^2/\Gamma_{4n}$.

Question. Find the optimal constant $C(g) > 0$ such that for any translation surface of genus g with a single singularity, we have $\mathrm{KVol}(X) > C(g)$.

We already know from [MM14] that $C(g) \geq 1$, and from [CKM21b] that $C(2) \leq 2$. In [Bou23b], I generalize [CKM21b] result to any genus, namely I show that $C(g) \leq g$. In fact, we conjecture from computer experiments that $C(g) = g$.

5. Other directions

Apart from the work presented here, I am also eager to work on other projects. I especially participated in several conferences and workshops of the ANR Adyct on spectral analysis on surfaces, as well as a reading group on Teichmüller theory and pseudo-Anosov diffeomorphisms. I also enjoy working with Sagemath to test my questions on simple examples and develop intuition. For example, I recently contributed to the implementation of the algebraic intersection of saddle connections on the package Flatsurf developped by V. Delecroix, J. Rüth and P. Hooper.

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