Hecke continued fractions and connection points on Veech surfaces

Julien Boulanger
Centro de Modelamiento Matematico, U. de Chile
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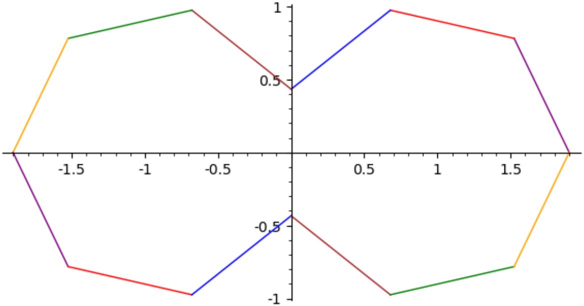
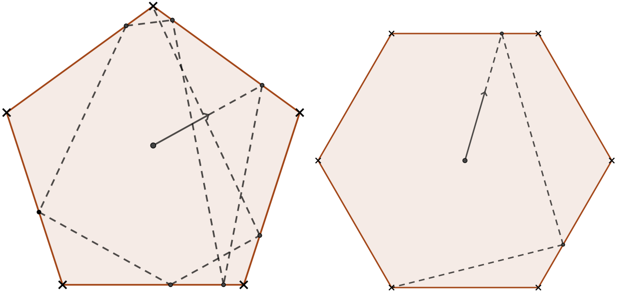


We consider an ideal billard trajectory on a regular polygon starting at the very center of the polygon.



Question

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Introduction: billiards in regular polygons

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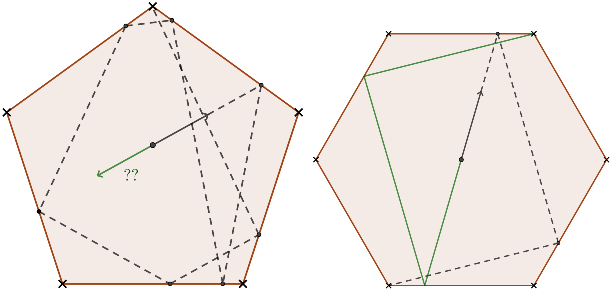

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- In a regular pentagon, the answer is YES, as a consequence of the work of A. Leutbecher, 1967, combined with results of W. Veech, 1989.

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- On a heptagonal and a nonagonal billiard ($n = 7, 9$), the answer is NO. (B.-2022)
- For $n \geq 11$ odd, the anwser is again NO. (B.-2025)

Outline

1. Hecke groups and cusp representatives
2. From billiards on regular polygons to Hecke groups
   - Unfolding a billard trajectory in a rational polygon
   - Saddle connections and connection points
   - Symmetries of a translation surface: the Veech group
3. Connection points

Hecke groups

Definition

The Hecke group H_n of level n is the subgroup of $PSL_2(\mathbb{R})$ generated by

$$S = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T_n = \pm \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix}$$

where $\lambda_n = 2\cos\frac{\pi}{n}$.

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→ S is a symmetry of order two: $S^2 = \pm I_2$.

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→ $H_3 = PSL_2(\mathbb{Z})$
→ S is a symmetry of order two: $S^2 = \pm I_2$.
→ We have

$$U_n := T_n S = \pm \begin{pmatrix} \cos\frac{\pi}{n} & \sin\frac{\pi}{n} \\ -\sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix}$$

and in particular $U_n^n = \pm I_2$.
→ More, the group H_n is the free product generated by U_n and S.

$$H_n = C_2 * C_n$$

The action on the hyperbolic plane

Proposition (Hecke)

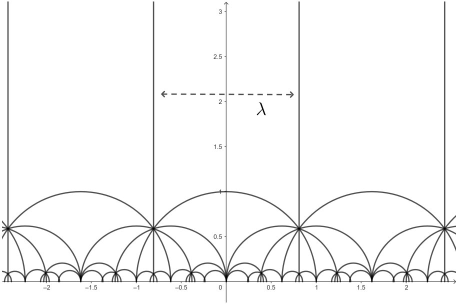
$H_n < PSL_2(\mathbb{R})$ *is a* ***discrete*** *subgroup, that is a* ***Fuchsian group***.

In particular, H_n acts on the hyperbolic plane by isometries (mobius transformations):

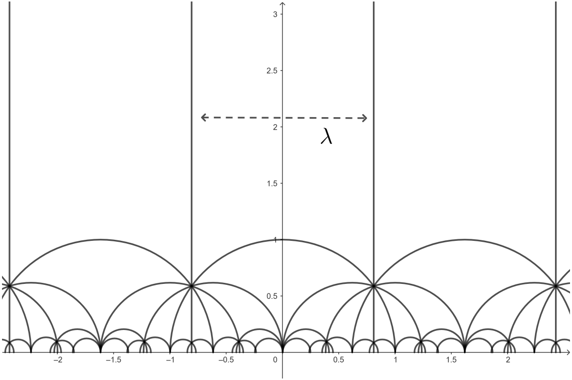
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

$$T \cdot z = z + \lambda_n$$

$$S \cdot z = \frac{-1}{z}$$



Rosen cusp challenge

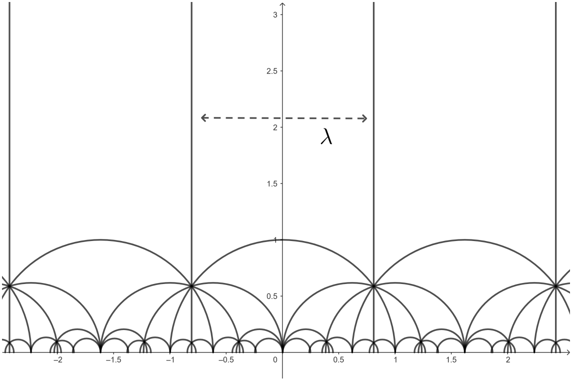


Question (Rosen cusp Challenge, 1954)

Determine $H_n \cdot \infty$.

In other words, we want to understand the set of **cusp representatives**, or **parabolic limit points**, i.e. fixed points of parabolic elements of H_n.

Rosen cusp challenge



Question (Rosen cusp Challenge, 1954)

Determine $H_n \cdot \infty$.

Today, we will restrict ourselves to the case where n is **odd**.

A first obstruction

Since $H_n \subset PSL_2(\mathbb{Z}[\lambda_n])$, we have, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_n$,

$$M \cdot \infty = \frac{a \cdot \infty + b}{c \cdot \infty + d} = \frac{a}{c} \in \mathbb{Q}[\lambda_n] \cup \{\infty\}$$

In other words:

$$H_n \cdot \infty \subseteq \mathbb{Q}[\lambda_n] \cup \{\infty\}$$

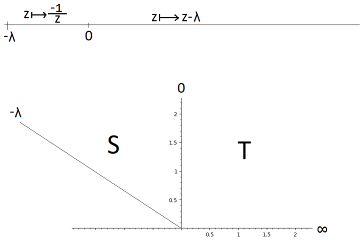
Hecke continued fractions

If $M = T^{a_0} S T^{a_1} \cdots S T^{a_k} S$, then

$$M \cdot \infty = a_0 \lambda - \cfrac{1}{a_1 \lambda - \frac{1}{a_2 \lambda - \dots}}$$

As a consequence, the elements of $H_n \cdot \infty$ are the real numbers having a finite *Hecke continued fraction expansion*.

A continued fraction expansion can be computed using the (Hecke) next-integer algorithm. More, this algorithm selects the finite expansion (if it exists).



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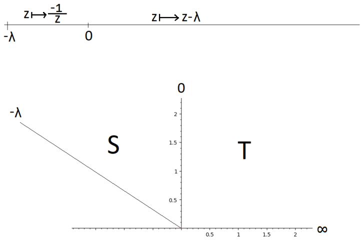

For $n = 3$, we have $\lambda_3 = 1$, $\mathbb{Q}[\lambda_3] = \mathbb{Q}$, and

$$H_3 \cdot \infty = PSL_2(\mathbb{Z}) \cdot \infty = \mathbb{Q} \cup \{\infty\}.$$

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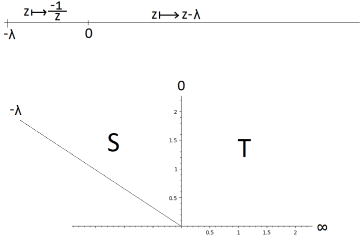

Theorem (Leutbecher, 1967)

$$H_5 \cdot \infty = \mathbb{Q}[\lambda_5] \cup \{\infty\} = \mathbb{Q}[\sqrt{5}] \cup \{\infty\}.$$

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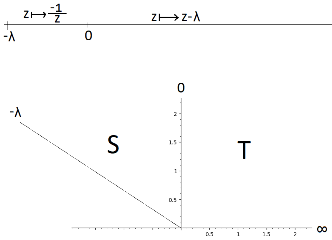

Theorem (Borho-Rosenberger, 1973 and Seibold, 1985)

For $n \geq 7$ odd,

$$H_n \cdot \infty \subsetneq \mathbb{Q}[\lambda_n] \cup \{\infty\}.$$

Eventually periodic Hecke continued fraction expansion

Theorem (Schmidt-Sheingorn, 1995)

The real numbers having an eventually periodic Hecke continued fraction expansion are the fixed directions of hyperbolic elements of H_n.

→ Such numbers always lie in a quadratic extension of $\mathbb{Q}[\lambda_n]$ (or in $\mathbb{Q}[\lambda_n]$ itself!)

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$\rightarrow$ For $n = 3, 5$, since every element of $\mathbb{Q}[\lambda_n]$ has a finite Hecke c.f., there is no hyperbolic fixed point in $\mathbb{Q}[\lambda_n]$.
$\rightarrow$ For $n = 7, 9$, we have $[\mathbb{Q}[\lambda_n] : \mathbb{Q}] = 3$ and there **are** hyperbolic fixed points in $\mathbb{Q}[\lambda_n]$. More, we have

Conjecture

Let $n = 7, 9$. Then every element of $\mathbb{Q}[\lambda_n]$ is either a parabolic fixed point (finite c.f.) or a hyperbolic fixed point (eventually periodic c.f.).

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Let $n = 7, 9$. Then every element of $\mathbb{Q}[\lambda_n]$ is either a parabolic fixed point (finite c.f.) or a hyperbolic fixed point (eventually periodic c.f.).

→ For $n \geq 11$, it seems that there are **no** hyperbolic fixed point in $\mathbb{Q}[\lambda_n]$!

Obstruction modulo two

Theorem (Borho-Rosenberger, 1973 - Seibold, 1985)

For $n \geq 7$ odd,

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Proof ($n = 7, 9$) There are elements in $\mathbb{Q}[\lambda_n]$ whose Hecke c.f. is eventually periodic. They do not belong to $H_n \cdot \infty$.

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Proof ($n \neq 9$). Using the correspondence

$$x = \frac{s}{t} \in \mathbb{Q}[\lambda_n] \cup \{\infty\} \leftrightarrow [s : t] \in \mathbb{P}^1(\mathbb{Z}[\lambda_n])$$

we can consider the reduced orbit modulo two:

$$\overline{H_n} \cdot [\bar{1} : \bar{0}] \subseteq \mathbb{P}^1(\mathbb{Z}[\lambda_n]/2\mathbb{Z}[\lambda_n])$$

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Lemma (Borho, 1973 - Borho-Rosenberger, 1973)

For $n \geq 7$ odd, apart from $n = 9$, the inclusion is strict.

Recap

- $H_n \cdot \infty \subset \mathbb{Q}[\lambda_n] \cup \{\infty\}$
- For odd n, this inclusion is an equality if and only if $n = 3$ or $n = 5$.
- The elements of $H_n \cdot \infty$ are the real numbers having a finite *next-integer Hecke continued fraction expansion*.
- For $n = 7, 9$, there are elements of $\mathbb{Q}[\lambda_n]$ with an infinite eventually periodic continued fraction expansion: these elements do not belong to $H_n \cdot \infty$.
- Inside $\mathbb{Q}[\lambda_n]$, and from $n \geq 7$, with the exception of $n = 9$, there is an *obstruction modulo two* for an element to be in $H_n \cdot \infty$.

Recap

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- Inside $\mathbb{Q}[\lambda_n]$, and from $n \geq 7$, with the exception of $n = 9$, there is an *obstruction modulo two* for an element to be in $H_n \cdot \infty$.

Conjecture

For $n = 7$, the obstruction modulo two is the only obstruction, that is $x = \frac{s}{t} \in H_n \cdot \infty$ if and only if $[\bar{s} : \bar{t}] \in \overline{H_n} \cdot [\bar{1} : \bar{0}]$.

$\rightarrow$ This conjecture is not true for $n = 9$, and it also seems not to be true for $n \geq 11$ although no proof is known.

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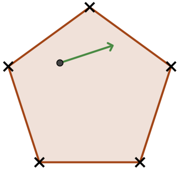
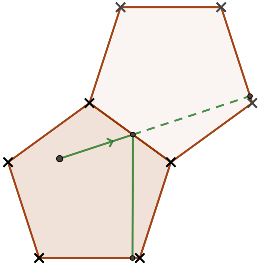
Unfolding a billiard trajectory in a rational polygon

We want to study a billard trajectory on the pentagon (resp. any *rational polygon*):



Unfolding a billiard trajectory in a rational polygon

When the trajectory reaches a side, instead of considering the usual billiard trajectory, we consider a reflected copy of the billiard table itself.
$\rightarrow$ The obtained mirror trajectory is then just a straight line!



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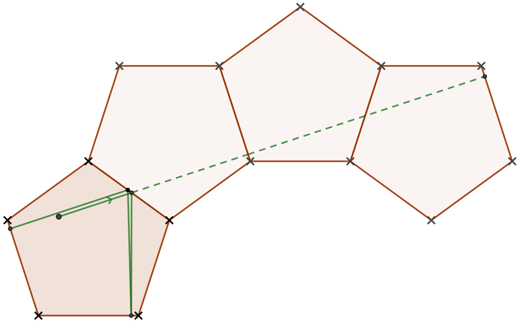
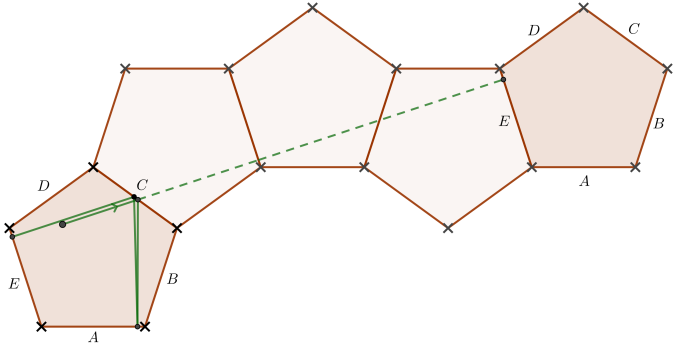


Unfolding a billiard trajectory in a rational polygon

At some point, we obtain a polygon which is a translated image of one of the previous polygons. We move it back to this previous polygon and we continue the trajectory there.



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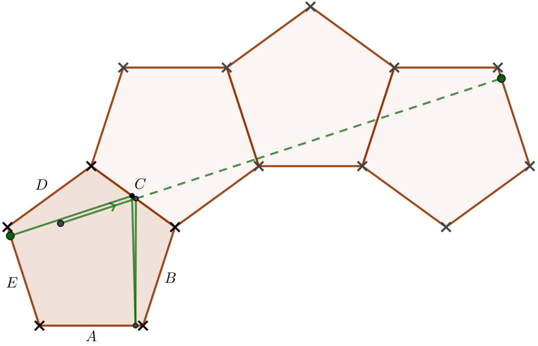
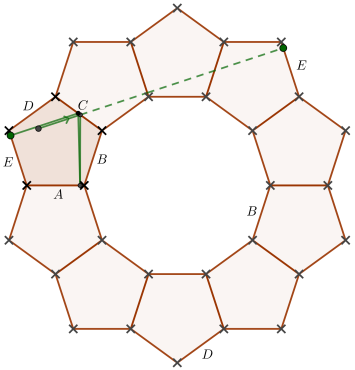


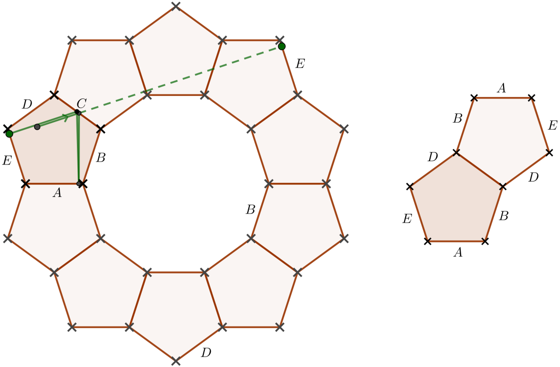
Unfolding a billiard trajectory in a rational polygon

Since there are only finitely many reflected copies of the pentagon (because its symmetry group is a **finite** dihedral group), this process transforms the trajectory in the original pentagonal billiard to a trajectory on a finite surface, given by polygons and identifications of sides and where **the identifications are given by translations**. Although the surface seems more complicated, the trajectory is now a straight line:



Unfolding a billiard trajectory in a rational polygon

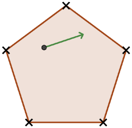
In fact, this necklace surface is a 5−cover of the **double regular pentagon** pictured below.



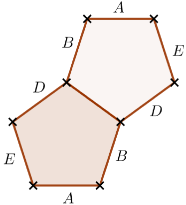
We will see that, for the purpose of studying trajectories it is sufficient to work on the double regular pentagon (resp. on the double regular n-gon for **odd** n).

Unfolding a billiard trajectory in a rational polygon

The study of the billiard flow in a rational polygon (i.e. whose angles are rational multiples of π) reduces to the study of the directionnal flow on a *translation surface*.



$\Longleftrightarrow$



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A translation surface is a surface obtained from a collection of euclidean polygon, by identifying pairs of parallel opposite sides of the same length (by translation).

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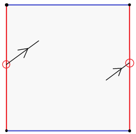
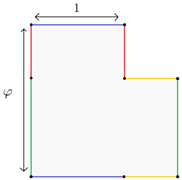
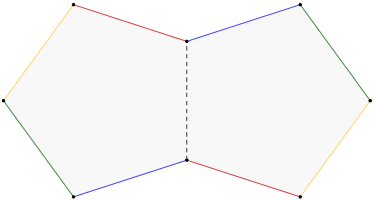




Golden L



Double pentagon

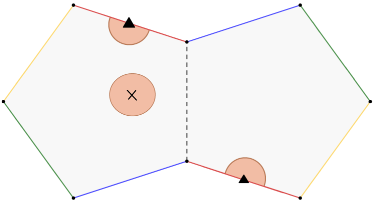
φ is the golden ratio

Translation surfaces

Definition

A translation surface is a surface obtained from a collection of euclidean polygon, by identifying pairs of parallel sides of the same length (by translation).

These surfaces can also be described as topological surfaces with an **atlas of charts** on the surface minus a finite set of points Σ such that all transition functions are translations, along with a distinguished direction.
Σ is the set of **singularities**.



Geodesics on a translation surface

Definition

A *separatrix* is a geodesic segment starting from a singularity.
A *saddle connection* is a geodesic segment from a singularity to a singularity.

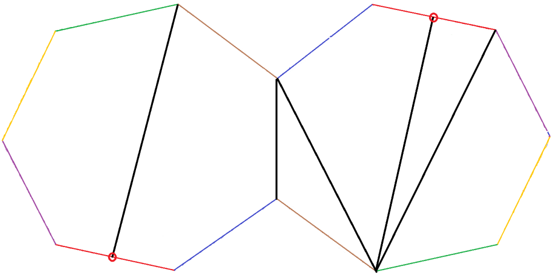


Figure: Examples of saddle connections on the double regular heptagon.

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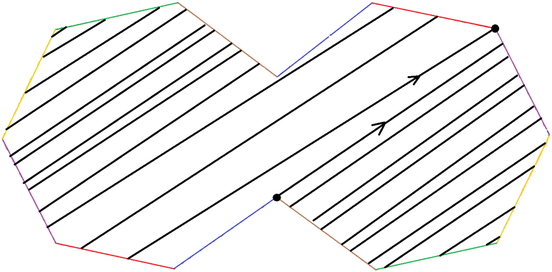

Figure: Another example of saddle connection on the double heptagon.

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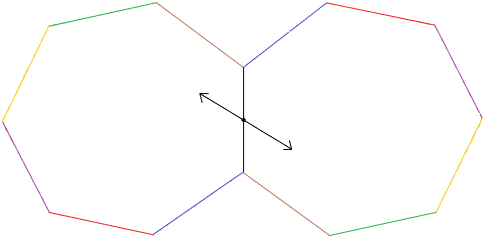
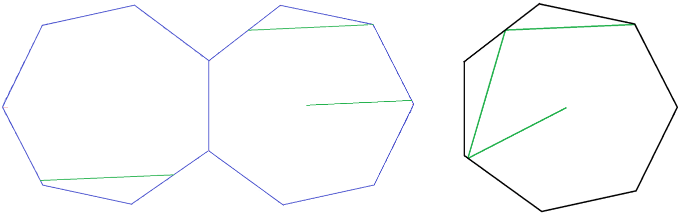
A *saddle connection* is a geodesic line from a singularity to a singularity.
A *connection point* is a non-singular point on a translation surface such that every separatrix passing through this point extends to a saddle connection.



The midpoints of the sides are connection points, because they are fixed points of an involution.

Theorem (B.- 2022 and 2025)

Given $n \geq 7$ odd, the centers of the n-gons are not connection points on the double regular n-gon.



An example of separatrix which does not extend to a saddle connection on the double heptagon, and the corresponding billiard trajectory on the regular heptagon.

Question

How to certify that a given separatrix does not extend to a saddle connection ?

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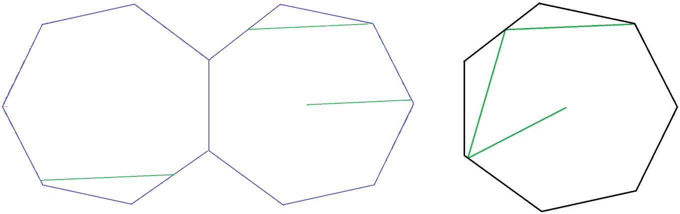
Given $n \geq 7$ odd, the centers of the n-gons are not connection points on the double regular n-gon.



An example of separatrix which does not extend to a saddle connection on the double heptagon, and the corresponding billiard trajectory on the regular heptagon.

Theorem (Veech, 1989)

The directions of saddle connections on the double regular n-gon are in bijection with the elements of $H_n \cdot \infty$.

1. Hecke groups and cusp representatives

2. From billiards on regular polygons to Hecke groups
   - Unfolding a billard trajectory in a rational polygon
   - Saddle connections and connection points
   - Symmetries of a translation surface: the Veech group

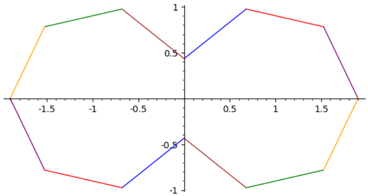
3. Connection points

Symmetries of a translation surface

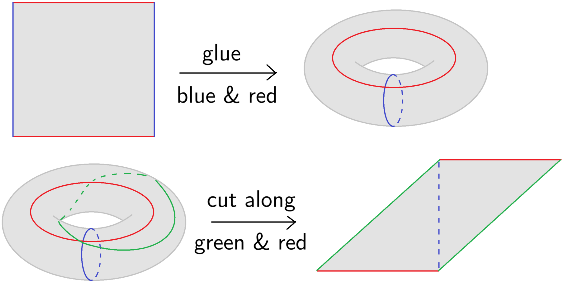
The double regular n-gon (for odd n) has a symmetry of order two and a symmetry of order n.

Definition

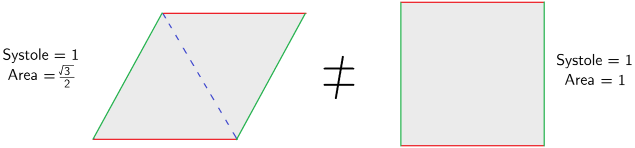
The **group of affine diffeomorphisms** of a translation surface X is the group of diffeomorphisms $\varphi : X \to X$ which are expressed as affine transformations in local charts.



The group of affine diffeomorphisms of the double regular n-gon is a free product $C_2 * C_n$.



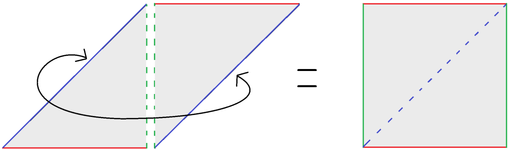
The two polygonal models above give the same resulting surface, whereas we have below two polygonal models of surfaces with different properties.



Definition

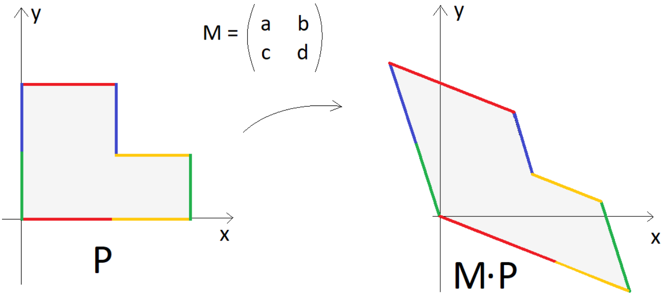
The **moduli space** $\Omega\mathcal{M}_g$ of translation surfaces of genus g is the set:

$$\Omega\mathcal{M}_g = \left\{ \begin{array}{c} \text{Collection of polygons with} \\ \text{identifications of parallel sides of} \\ \text{the same length and genus } g \end{array} \right\} / \text{cut and paste}$$



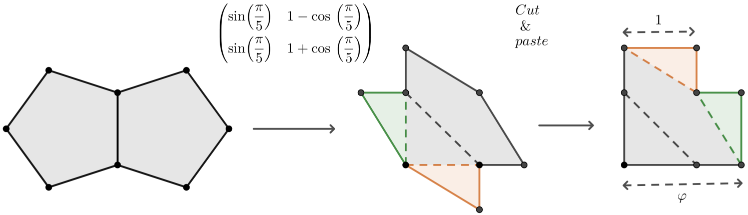
$SL_2(\mathbb{R})$-action

Given a translation surface X described by a collection of polygons and $M \in GL_2^+(\mathbb{R})$, we can construct the translation surface $M \cdot X$.



It is often convenient to consider the action of $SL_2(\mathbb{R})$ instead of $GL_2^+(\mathbb{R})$ as it preserves the area.

Example: the golden L and the double pentagon



Lemma

The double pentagon and the golden L belong to the same $GL_2^+(\mathbb{R})$ *orbit.*

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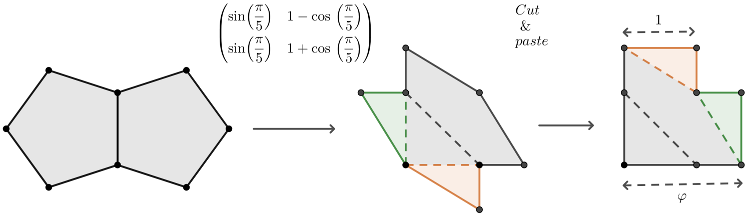


Lemma

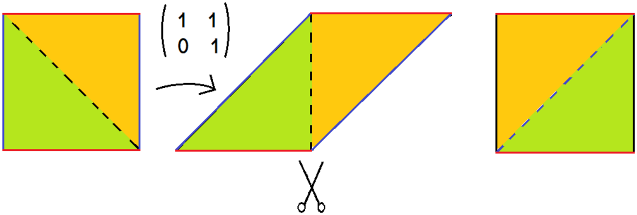
The double pentagon and the golden L belong to the same $GL_2^+(\mathbb{R})$ orbit.

$\rightarrow$ In particular, there is a bijective correspondence between saddle connections (resp. connection points) on these two surfaces !

Veech group

Definition

The Veech group of a translation surface X is the stabilizer of X (in the moduli space) under the action of $SL_2(\mathbb{R})$. We denote it by $SL(X)$.



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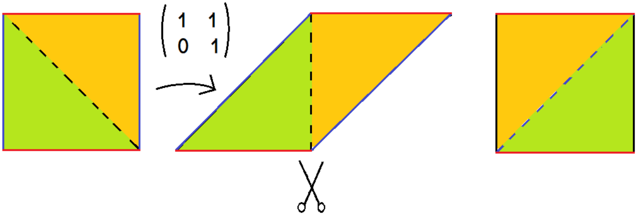

Proposition

The Veech group the flat square torus is $SL_2(\mathbb{Z})$.

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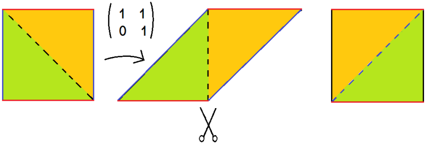
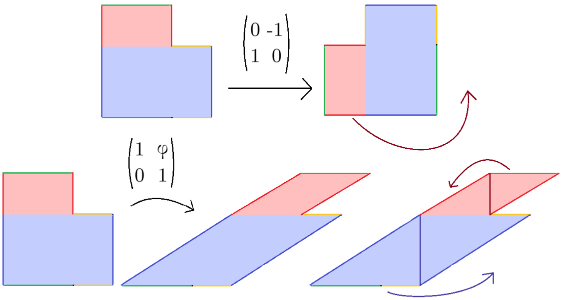


Theorem (W.Veech, 1989)

For any translation surface X, $SL(X)$ is a discrete subgroup of $SL_2(\mathbb{R})$.

Example : the golden L



Proposition

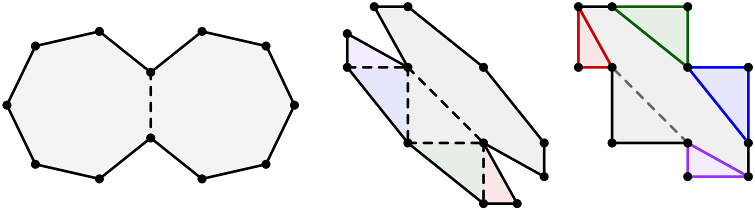
The Veech group of the golden L is the Hecke group H_5, generated by

$$S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } T := \begin{pmatrix} 1 & \varphi \\ 0 & 1 \end{pmatrix}$$

where $\varphi = 2\cos\frac{\pi}{5} = \lambda_5$ is the golden mean.

The staircase model of the double regular n-gon

More generally, to a double regular n gon is associated a staircase model:



From the double regular n-gon to its staircase model, we first apply the matrix

$$P = \frac{1}{\sin\frac{(n-1)\pi}{2n}} \begin{pmatrix} \sin\frac{\pi}{n} & -\cos\frac{\pi}{n}+1 \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n}+1 \end{pmatrix}.$$

then we cut and paste the pieces to obtain a staircase surface.

Theorem (Veech, 1989)

Let $n \geq 3$ odd. The Veech group of the staircase model of the double regular n-gon is the Hecke group of level n.

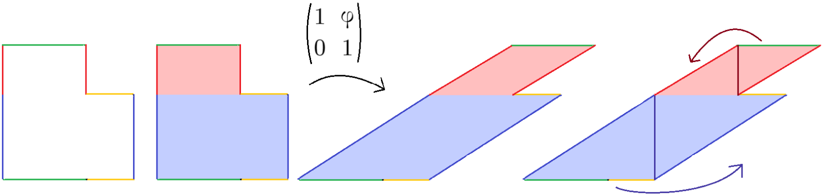
Periodic directions on Veech surfaces

Theorem (Veech, 1989)

Let X be a translation surface whose Veech group $SL(X)$ is a lattice. Given a direction θ on X, we have the following alternative:

- *either the flow in direction θ is completely periodic (all geodesics in this direction are closed or saddle connections)*
- *or it is uniquely ergodic (geodesics in this direction equidistribute with respect to the lebesgue measure).*

Further, the first case arise if and only if θ is an eigendirection of a parabolic matrix of $SL(X)$.



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Corollary

Periodic directions on the staircase model of the double regular n-gon are those whose slope belong to $H_n \cdot \infty$.

Connection points: a few motivations

Question

What are the Fuchsian groups that arise as Veech groups of translation surfaces ?

Theorem (Hubert-Schmidt, 2004)

Let X be a translation surface whose Veech group is a lattice. Let P be a connection point on X such that its orbit under the action of the affine group is not finite. Then any surface X_P constructed as a finite (translation) cover of X ramified at P has a Veech group which is not finitely generated.

Theorem (Consequence of McMullen, 2006)

When the algebraic field K_X generated by $\{tr(M), M \in SL(X)\}$ is $\mathbb{Q}$ or quadratic over $\mathbb{Q}$, we have a complete characterisation of connection points.

Connection points: a few motivations

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The case $[K_X : \mathbb{Q}] \geq 3$

When the algebraic field K_X generated by $\{tr(M), M \in SL(X)\}$ has degree three or more over $\mathbb{Q}$, we do not know any example of connection point whose orbit under the action of the affine group is not finite.

Recap

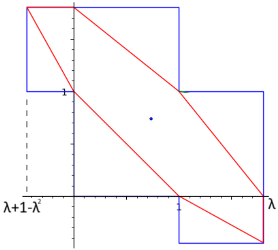
- We want to know whether or not the central points of the double regular n-gon are connection points for $n \geq 3$ odd.
- We work in the associated staircase model instead, where directions of saddle connections are those with slope in $H_n \cdot \infty$.

Recap

- We want to know whether or not the central points of the double regular n-gon are connection points for $n \geq 3$ odd.
- We work in the associated staircase model instead, where directions of saddle connections are those with slope in $H_n \cdot \infty$.

First obstruction: $H_n \cdot \infty \subset \mathbb{Q}[\lambda_n] \cup \{\infty\}$.

$\rightarrow$ Connection points on the staircase model of the double regular n-gon have coordinates in $\mathbb{Q}[\lambda_n]$.

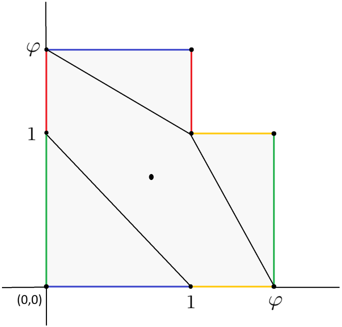


The points in the staircase model corresponding to central points have coordinates of the form $\frac{1}{n}(x, x)$ and $x \in \mathbb{Z}[\lambda_n]$.

Theorem (Leutbecher, 1967)

$$H_5 \cdot \infty = \mathbb{Q}[\lambda_5] \cup \{\infty\} = \mathbb{Q}[\sqrt{5}] \cup \{\infty\}.$$

As a consequence, the central points of the double pentagon are connection points.
(As well as any point whose coordinates in the golden L lie in $\mathbb{Q}[\sqrt{5}]$.)



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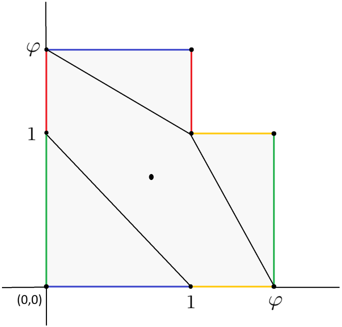


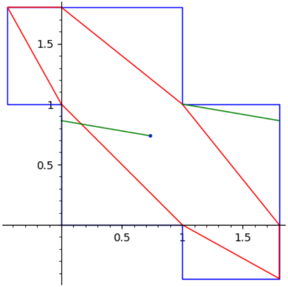
Theorem (Borho-Rosenberger, 1973 - Seibold, 1985)

For $n \geq 7$ odd,

$$H_n \cdot \infty \subsetneq \mathbb{Q}[\lambda_n] \cup \{\infty\}.$$

$n = 7$ and $n = 9$

For $n = 7, 9$, we can use Hecke continued fractions:

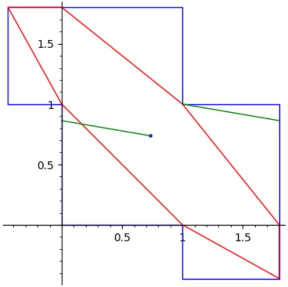


For $n = 7$, the central point is not a connection point:
Here is an example of a separatrix passing through one of the central point whose slope has an eventually periodic Hecke continued fraction expansion. (B.-2022).

For $n = 9$, the same argument works, you can find (easily) an explicit separatrix whose slope has an eventually periodic Hecke continued fraction expansion (B.-2022).

$n = 7$ and $n = 9$

For $n = 7, 9$, we can use Hecke continued fractions:



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For $n \geq 11$, this argument does not work anymore... As we said, we conjecture that they are **no** elements of $\mathbb{Q}[\lambda_n]$ whose Hecke continued fraction expansion is eventually periodic !

Using the obstruction modulo two

Theorem (B.- 2025)

Let $n \geq 7$ odd, $n \neq 9$. Let P a point on the staircase model of the double regular n-gon whose coordinates are of the form $\frac{1}{N}(x, y)$ with $x, y \in \mathbb{Z}[\lambda]$ and $N \in \mathbb{N}^\star$. Then,

- *If $N \geq 1$ is odd, then P is not a connection point.*
- *If N is even and $[\overline{x}, \overline{y}] \notin \overline{H_n} \cdot [\overline{1} : \overline{0}]$, then P is not a connection point.*

Corollary

For $n \geq 7$ odd, $n \neq 9$, the central points of the double regular n-gon are not connection points.

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Corollary

For $n \geq 7$ odd, $n \neq 9$, the central points of the double regular n-gon are not connection points.

Question

Apart from the middle of the sides, does there exist connection points on the double regular n-gon for $n \geq 7$ odd ?

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Conjecture

- For $n = 7$, all the remaining points are connection points.

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Question

Apart from the middle of the sides, does there exist connection points on the double regular n-gon for $n \geq 7$ odd ?

Conjecture

- For $n = 7$, all the remaining points are connection points.
- For $n \geq 9$, the middle of the sides are the only connection points.

Theorem (B.- 2025)

Let $n \geq 7$ odd, $n \neq 9$. Let P a point on the staircase model of the double regular n-gon whose coordinates are of the form $\frac{1}{N}(x, y)$ with $x, y \in \mathbb{Z}[\lambda]$ and $N \in \mathbb{N}^\star$. Then,

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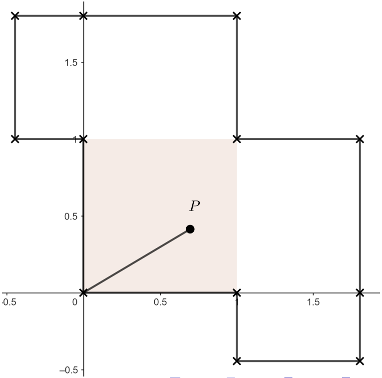
- *If N is even and $[\overline{x}, \overline{y}] \notin \overline{H_n} \cdot [\overline{1} : \overline{0}]$, then P is not a connection point.*

Idea of the proof: if P lies in the central square and $[\overline{x} : \overline{y}] \notin \overline{H_n} \cdot [\overline{0} : \overline{1}]$, then we can consider the separatrix from the origin to P: its slope is $[x : y]$ which is thus not in a periodic direction.



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- *If N is even and $[\overline{x}, \overline{y}] \notin \overline{H_n} \cdot [\overline{1} : \overline{0}]$, then P is not a connection point.*

Idea of the proof: If, for example, P lies in the top horizontal cylinder, then

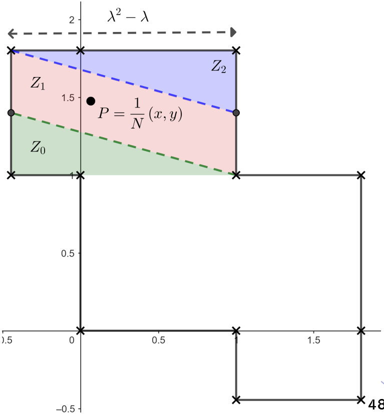
$$\varphi_H^2(P) = \tfrac{1}{N}(x + 2(y - N)\lambda - \varepsilon N(\lambda^2 - \lambda), y)$$

where $\varepsilon \in \{0, 1, 2\}$ corresponds to the zone Z_0, Z_1, Z_2 of P. Hence, the reduction modulo two of the coordinates of $\varphi_H^2(P)$ is

$$(\overline{x} + \overline{\varepsilon(\lambda^2 - \lambda)}, \overline{y})$$

if N is odd

(and it is left unchanged if N is even).



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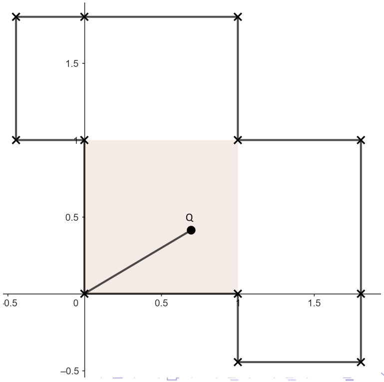
Idea of the proof: Playing with twists along cylinders, we are able to construct a point Q in the orbit of P under the action of the group of affine diffeomorphism of X which lies in the central square and for which the separatrix from the origin to Q is not in a periodic direction.

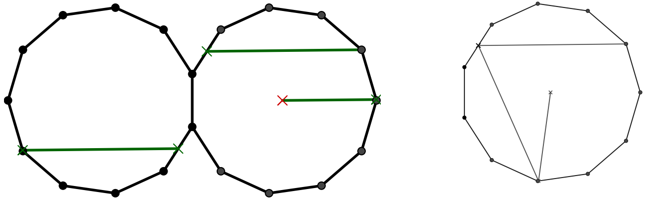


Theorem (B. - 2025)

*Let $n \geq 7$ be a **prime number**. We consider the double regular n-gon with the origin placed at the center of the right n-gon. Then the separatrix starting from the point of coordinates $\left(\cos \frac{2\pi}{n}, \sin \frac{2\pi}{n}\right)$ with direction*

$$(X, Y) = \left(1 + 2\left(1 + \cos\frac{\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right), 2\left(1 - \cos\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\right),$$

passes through the origin (the central point of the right n-gon) and does not extend to a saddle connection.



→ This uses elementary computations in $\mathbb{Q}[2\cos\frac{\pi}{n}]$.

Thanks for your attention !