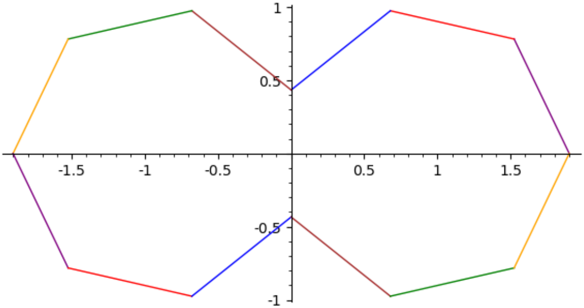
Algebraic intersections in translation surfaces

Julien Boulanger

20 December 2023



Outline

1. Translation surfaces and their Veech groups

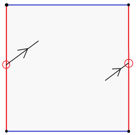
2. The algebraic interaction strength KVol

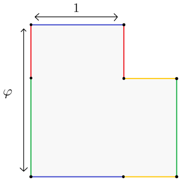
3. A few geometric ideas

Translation surfaces

Definition

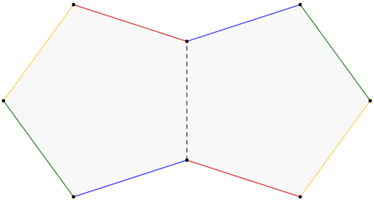
A translation surface is a surface obtained from a collection of euclidean polygon, by identifying pairs of parallel opposite sides of the same length (by translation).





Golden L

Double pentagon



φ is the golden ratio

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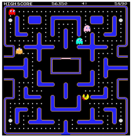

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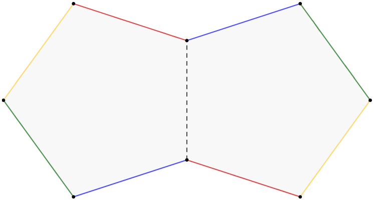

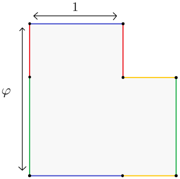

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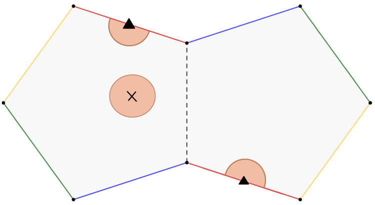
These surfaces can also be described as topological surfaces with an **atlas of charts** on the surface minus a finite set of points Σ such that all transition functions are translations, along with a distinguished direction.
Σ is the set of **singularities**.



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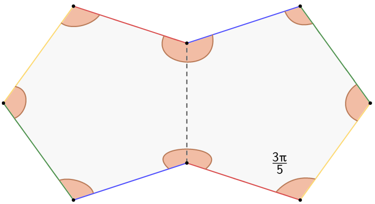
In the example of the double pentagon on the right, the vertices of the two pentagons are all identified to the same point on the resulting surface. This point is a **conical singularity** of angle 6π.



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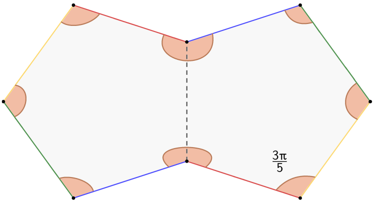
In a translation surface, the singularities are always conical singularities of angle $2k\pi, k \geq 2$.

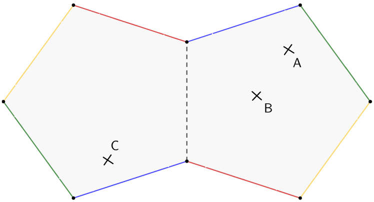


Translation surfaces

Definition

A translation surface is a surface obtained from a collection of euclidean polygon, by identifying pairs of parallel sides of the same length (by translation).

The surface inherits the flat structure from $\mathbb{R}^2$ outside the singularities. We have notions of length, geodesics and angles.



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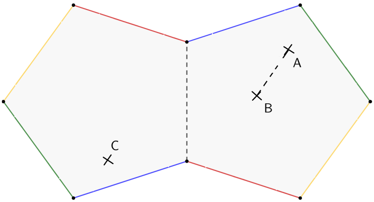

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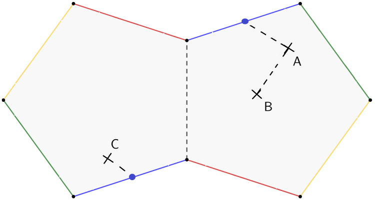
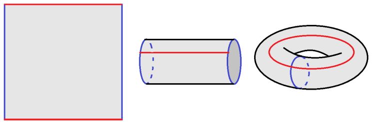
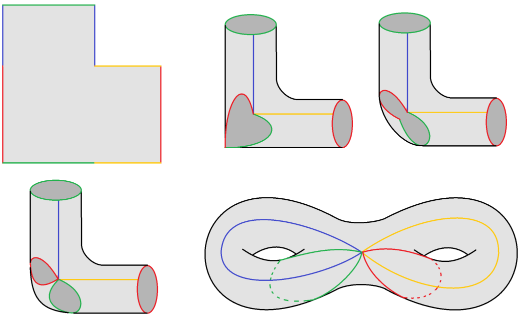


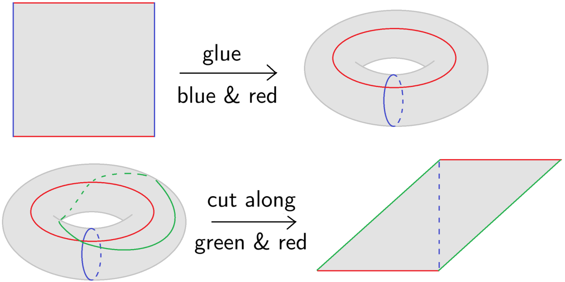
Topologically, you can glue the sides of the polygon to make an actual surface.



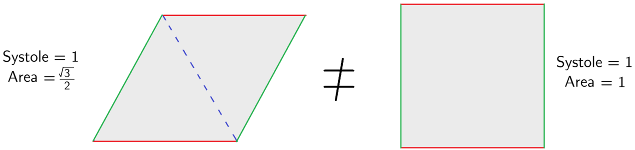
The **genus** is the *number of holes* of the corresponding glued surface.



Teichmüller space and moduli space



The two polygonal models above give the same resulting surface, whereas we have below two polygonal models of surfaces with different properties.

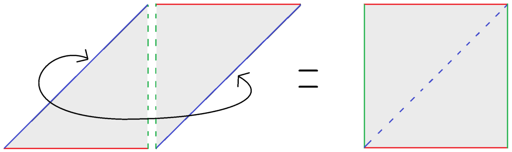


Definition

The **moduli space** $\Omega\mathcal{M}_g$ of translation surfaces of genus g is the set:

$$\Omega\mathcal{M}_g = \left\{ \begin{array}{c} \text{Collection of polygons with} \\ \text{identifications of parallel sides of} \\ \text{the same length and genus } g \end{array} \right\} / \text{cut and paste}$$

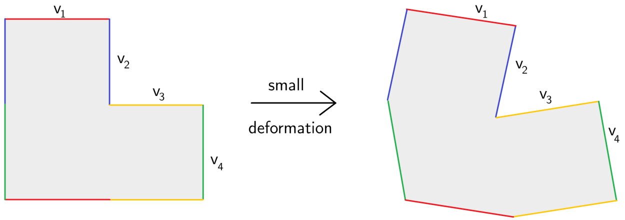
The **Teichmüller space** $\Omega\mathcal{T}_g$ of translation surfaces can be seen as the space of (X, φ) where $X \in \Omega\mathcal{M}_g$ and φ is a *marking* of a homology basis.



We have $\Omega\mathcal{M}_g = \Omega\mathcal{T}_g / MCG(g)$.

Moving in the Teichmüller space : local coordinates

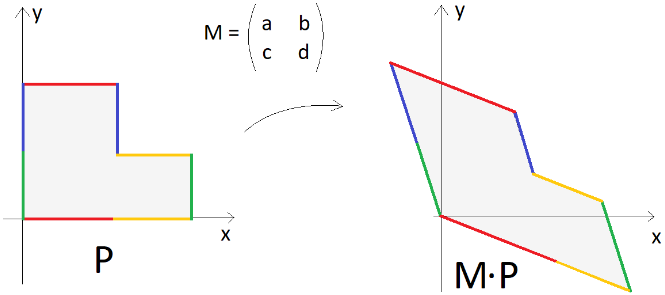
The Teichmüller (resp. Moduli) space of translation surfaces has a natural geometry. Namely if two translation surfaces can be obtained from one to another by a *"small"* deformation, these surfaces will be *"close"* to each other in the Teichmüller space.



In this example, the vectors v_1, v_2, v_3 and v_4 give local coordinates to the Teichmüller space around this L-shaped translation surface.

$SL_2(\mathbb{R})$-action

Given a translation surface X described by a collection of polygons and $M \in GL_2^+(\mathbb{R})$, we can construct the translation surface $M \cdot X$.

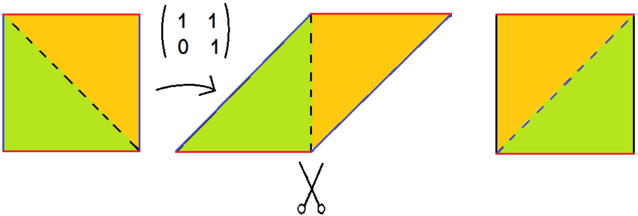


It is often convenient to consider the action of $SL_2(\mathbb{R})$ instead of $GL_2^+(\mathbb{R})$ as it preserves the area.

Veech group

Definition

The Veech group of a translation surface X is the stabilizer of X (in the moduli space) under the action of $SL_2(\mathbb{R})$. We denote it by $SL(X)$.



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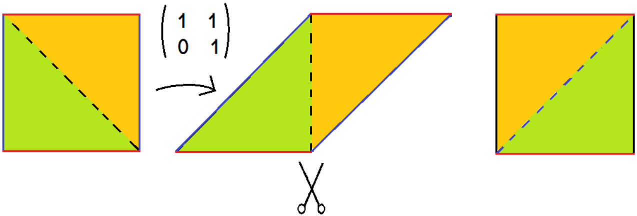

Proposition

The Veech group of any flat torus is conjugated to $SL_2(\mathbb{Z})$.

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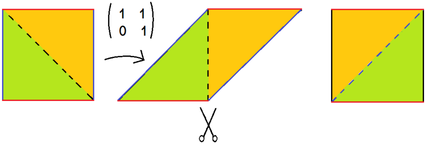

Theorem (W.Veech, 1989)

For any translation surface X, $SL(X)$ is a discrete subgroup of $SL_2(\mathbb{R})$.

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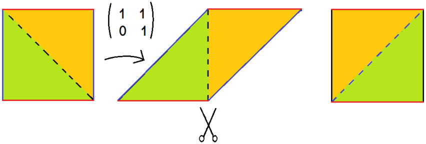


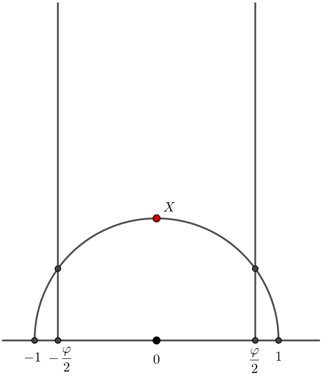
Theorem (W.Veech, 1989)

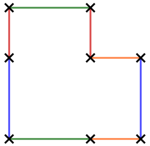
For any translation surface X, $SL(X)$ is a discrete subgroup of $SL_2(\mathbb{R})$.

Consequence: If we quotient by the action of the rotations $SO_2(\mathbb{R})$, the orbit of X under the action of $SL_2(\mathbb{R})$ can be identified with $\mathbb{H}^2/SL(X)$.

Example : The golden L

We choose a base surface S in the orbit. If $X = M \cdot S$ with $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, we map the surface X to the point $\frac{di+b}{ci+a} \in \mathbb{H}^2$.





$t = 0$ $\qquad M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$s = 0$ $\qquad X = (0, 1)$

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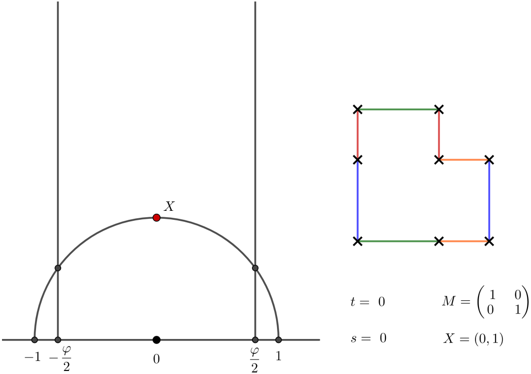

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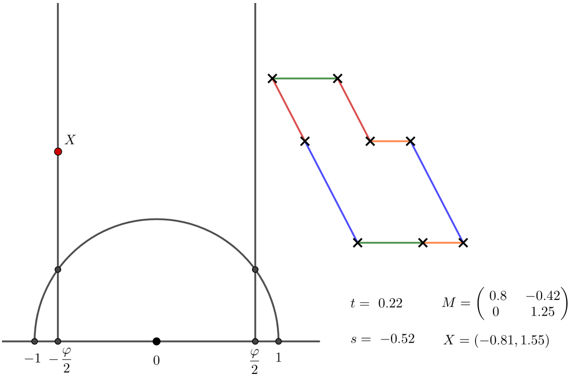

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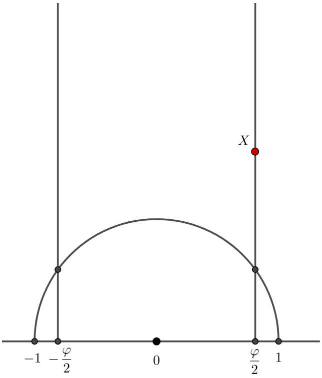

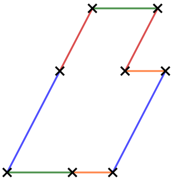

$t = 0.22$ $\quad M = \begin{pmatrix} 0.8 & 0.42 \\ 0 & 1.25 \end{pmatrix}$

$s = 0.52$ $\quad X = (0.81, 1.55)$

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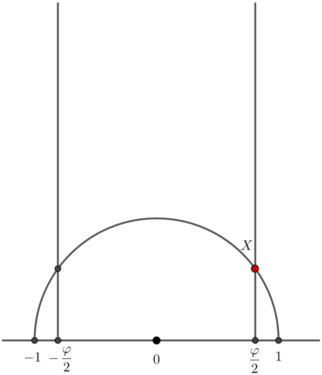

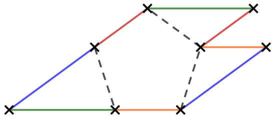

$t = \ -0.27 \qquad M = \begin{pmatrix} 1.3 & 1.79 \\ 0 & 0.77 \end{pmatrix}$

$s = \ 1.37 \qquad X = (0.81, 0.59)$

Example : The golden L

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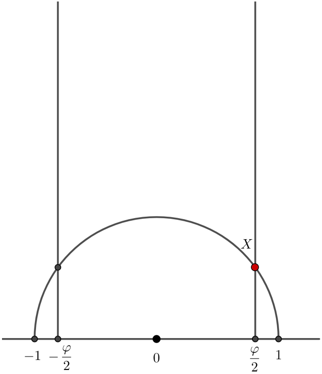
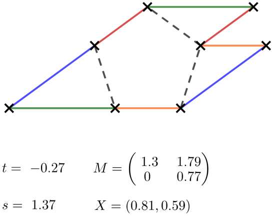
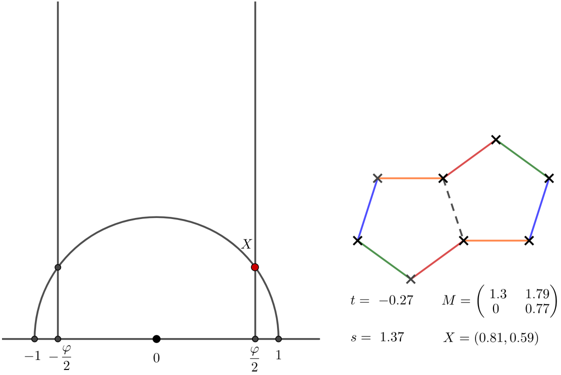




Example : The golden L

Proposition

The golden L and the double pentagon belong to the same $GL_2^+(\mathbb{R})$-orbit.
The Veech group of the golden L is the triangle group $\Delta^+(2, 5, \infty)$.



Veech surfaces and cylinder decompositions

Definition

A **Veech surface** is a translation surface whose Veech group is a lattice.

Theorem (Veech, 1989)

Each direction on a Veech surface is either uniquely ergodic or completely periodic (all geodesics in this direction are closed).

It implies the existence of a cylinder decomposition in every periodic direction.

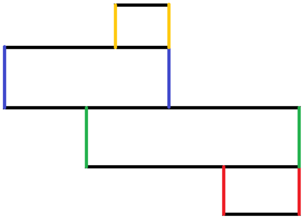


Figure: A translation surface decomposed into horizontal cylinders.

Bouw-Möller surfaces

Given $m, n \geq 2$ with $mn \geq 6$, the Bouw-Möller surface $S_{m,n}$ is a translation surface build from a collection of m polygons $P_0, \cdots, P_{m-1}$ such that:

- P_0 and P_{m-1} are regular n-gons of side length $\sin(\frac{\pi}{m})$,
- $\forall k \in \{2, \cdots, m-2\}$, P_k is an equiangular $2n$-gon, and its sides has length alternating between $\sin(\frac{k\pi}{m})$ and $\sin(\frac{(k+1)\pi}{m})$,
- Sides of P_k are alternatively glued to sides of P_{k-1} and P_{k+1}.

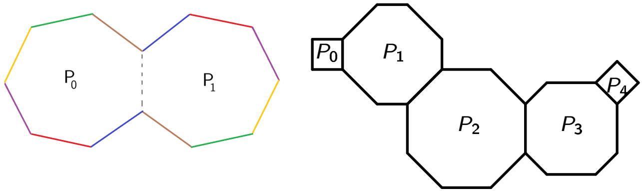


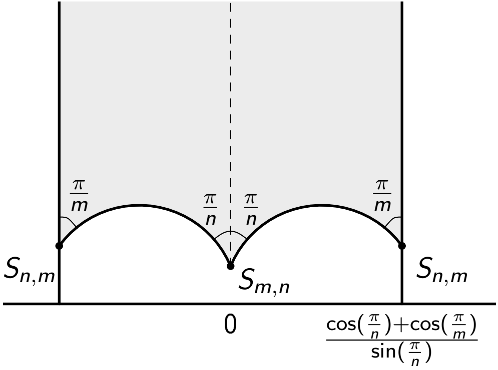
Figure: The Bouw-Möller surfaces $S_{2,7}$ (*left*) and $S_{5,4}$ (*right*).

Proposition

The surface $S_{m,n}$ has $\gamma = \gcd(m,n)$ singularities and genus $\frac{mn-m-n-\gamma}{2}$.

Theorem

The Veech group of $S_{m,n}$ is the triangle group $\Delta^+(m,n,\infty)$.



A fundamental domain in $\mathbb{H}^2$ for $\Delta^+(m,n,\infty)$.

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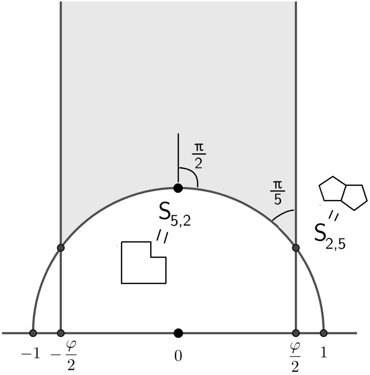


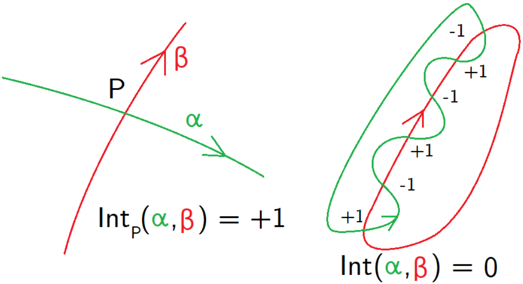
For $m = 2$ and $n = 5$, we get back to the double pentagon.
For $m = 5$ and $n = 2$, we get back to the golden L. (for $n = 2$, the we have to consider a degenerate polygon with two sides)

Outline

Algebraic intersection

Let X be a closed oriented surface with a Riemannian metric, possibly with singularities.

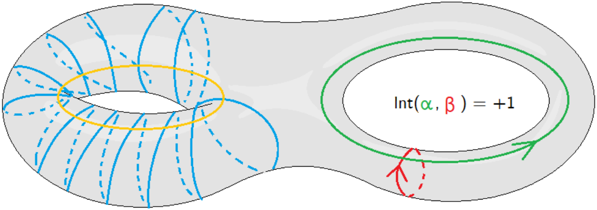
Given two oriented closed curves α and β on X, consider the algebraic intersection Int(α, β).



The algebraic intersection $\mathrm{Int}(\cdot,\cdot)$ is a bilinear symplectic form on homology.

Question

How many times can two closed curves of a given length intersect ?



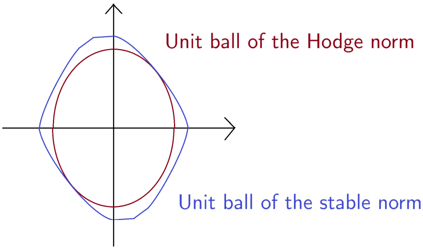
Definition

$$\mathrm{KVol}(X) := \mathrm{Vol}(X) \cdot \sup_{\alpha,\beta \text{ closed curves}} \frac{\mathrm{Int}(\alpha,\beta)}{l(\alpha)l(\beta)}$$

Remark : Multiplying by the volume makes KVol scalar invariant.

→ In D. Massart's thesis (1996), KVol arises as a comparison constant between the *stable norm* $\|\cdot\|_s$ and the *Hodge norm* $\|\cdot\|_2$ in homology, namely we have for all $h \in H_1(X, \mathbb{R})$,

$$\frac{1}{\sqrt{\mathrm{Vol}(X)}}\|h\|_s \le \|h\|_2 \le \mathrm{KVol}(X)\frac{1}{\sqrt{\mathrm{Vol}(X)}}\|h\|_s.$$



The Hodge norm (coming from the L^2 norm in cohomology) is euclidean: its unit ball is an ellipse.

The stable norm depends on the metric, its unit ball can be very complicated (e.g. polygonal with an infinite number of sides)

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$$\frac{1}{\sqrt{\mathrm{Vol}(X)}}\|h\|_s \leq \|h\|_2 \leq \mathrm{KVol}(X)\frac{1}{\sqrt{\mathrm{Vol}(X)}}\|h\|_s.$$

→ In 2014, D. Massart and B. Muetzel studied the behaviour of KVol(X) as X goes towards the boundary of the moduli space of hyperbolic surfaces, and gave geometric bounds on KVol, namely for any Riemannian surface:

$$\frac{\mathrm{Vol}(X)}{2Dl_0} \leq \mathrm{KVol}(X) \leq 9\frac{\mathrm{Vol}(X)}{l_0^2} = 9 \cdot \mathrm{SysVol}(X),$$

where D is the diameter and l_0 the homological systolic length of X.

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$$\frac{1}{\sqrt{\mathrm{Vol}(X)}}\|h\|_s \leq \|h\|_2 \leq \mathrm{KVol}(X)\frac{1}{\sqrt{\mathrm{Vol}(X)}}\|h\|_s.$$

Theorem (Massart, Muetzel, 2014)

For every Riemannian surface X of genus $g \geq 1$, we have:

$$KVol(X) \geq 1$$

with equality if and only if X is a flat torus.

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Theorem (Balacheff, Karam, Parlier, 2021)

There exist $c > 0$ such that for any hyperbolic surface X of genus $g \geq 2$, we have

$$KVol(X) \geq c\frac{g}{(log(g))^2}$$

This growth rate is optimal, see [Buser, Sarnak 1994].

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This growth rate is optimal, see [Buser, Sarnak 1994].

Question

What about the growth rate of KVol with the genus on translation surfaces ?

Conjecture

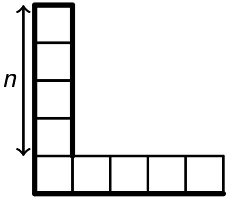
For any translation surface X of genus $g \geq 1$ with a single singularity,

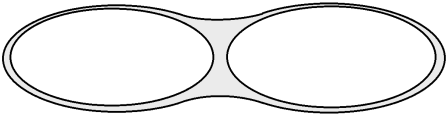
$$KVol(X) > g$$

Theorem (Cheboui, Kessi, Massart, 2021)

There exists a family $(L_n^{(2)})_{n\in\mathbb{N}}$ of translation surfaces of genus 2 such that

$$KVol(L_n^{(2)}) \underset{n\to+\infty}{\longrightarrow} 2$$





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Theorem (B. 2023)

For any $g \geq 2$, there exists a family $(L_n^{(g)})_{n\in\mathbb{N}}$ of translation surfaces of genus g with a single singularity such that

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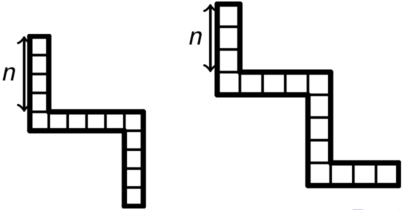
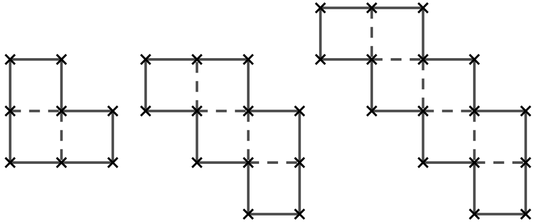


Question

Is it possible to compute explicitely KVol on some examples of translation surfaces ?

- We have seen that KVol is 1 on a flat torus.
- In 2021, S. Cheboui, A. Kessi and D.Massart provide a method to compute KVol on $SL_2(\mathbb{R})$-orbits of a family of squared tiled staircase surfaces.



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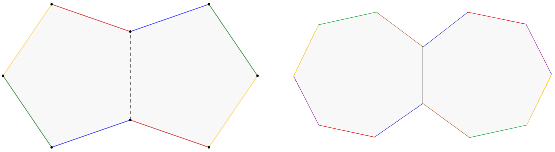

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- In 2022, we extend this method with E.Lanneau and D.Massart to compute KVol on the $SL_2(\mathbb{R})$-orbit of the double regular n-gons for odd n.
- In 2023, we deal with the case of the regular n-gon for even n.
- We then generalize the method with I. Pasquinelli to the case of Bouw-Möller surfaces with a single singularity.

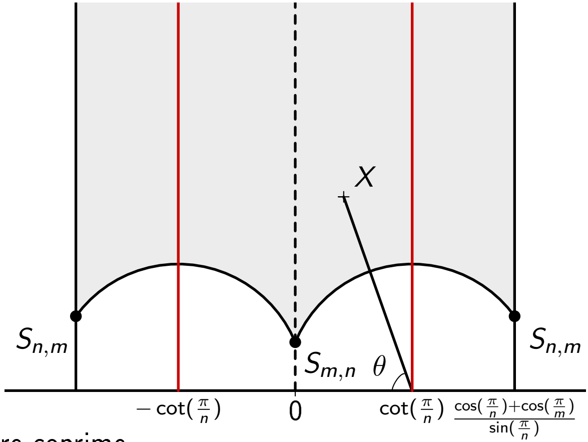
Theorem (B.- Pasquinelli)

*Let $m, n \geq 2$ **coprime**. For any surface $X = M \cdot S_{m,n}$ obtained from $S_{m,n}$ by the action of a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, we have:*

$$\mathrm{KVol}(X) = K_{m,n} \cdot \frac{1}{\cosh\left(d_{\mathbb{H}^2}\left(\Gamma_{m,n} \cdot \frac{di+b}{ci+a}, \gamma_{\infty, \pm \cot \pi/n}\right)\right)}$$

where

- *$K_{m,n} > 0$ is an explicit constant which only depends on m and n,*
- *$\gamma_{\infty, \pm \cot \pi/n}$ is the union of the two hyperbolic geodesics of respective endpoints $\pm \cot \pi/n$ and ∞.*
- *$d_{\mathbb{H}^2}$ is the hyperbolic distance,*
- *$\Gamma_{m,n} < SL_2(\mathbb{R})$ is the Veech group of $S_{m,n}$.*



If m and n are coprime,

$$\mathrm{KVol}(X) = K_{m,n} \cdot \frac{1}{\cosh(d_{\mathbb{H}^2}(\Gamma_{m,n} \cdot \frac{di+b}{ci+a}, \gamma_{\infty,\pm\cot\pi/n}))} = K_{m,n} \sin\theta(X)$$

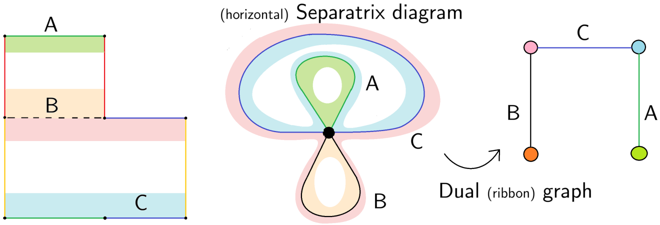
Related results

Theorem (B., 2023)

KVol is bounded on the $SL_2(\mathbb{R})$-orbit of a Veech surface if and only if there are no (algebraically) intersecting parallel goedesics.

Theorem (B.-Pasquinelli)

There are no intersecting parallel geodesics on Bouw-Möller surfaces. Hence, KVol is bounded on the $SL_2(\mathbb{R})$-orbit of any Bouw-Möller surface.



Theorem (B-Pasquinelli)

Let X be a translation surface constructed from a collection of polygons such that:

(H1) *All the polygons are **convex** with **obtuse angles**.*

(H2) *Sides of the same polygon are not paired together.*

Then:

$$\mathrm{KVol}(X) \leq \frac{\mathrm{Vol}(X)}{l_0^2}$$

with equality if and only if there are two sides of length l_0 that represent closed curves and intersect once at a singularity. Here l_0 is the length of the shortest side of the polygons.

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- Examples of such surfaces include the Bouw-Möller surfaces $S_{m,n}$ for $n \geq 5$, but a similar result holds for the regular $2n$-gon for $n \geq 8$, as well as the Bouw-Möller surfaces $S_{m,n}$ with $n = 2, 3$ or 4.
- For $X = S_{m,n}$ with coprime m, n, we get equality above.

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- If X has a single singularity, and all the vertices of the polygons are identified with the singularity, then l_0 is the homological systolic length of X. Compare with [Massart-Müetzel 2014]:

$$\mathrm{KVol}(X) \leq 9\frac{\mathrm{Vol}(X)}{l_0^2}.$$

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- If X has a single singularity, and all the vertices of the polygons are identified with the singularity, then l_0 is the systolic length of X.

Conjecture

For any translation surface X, $\mathrm{KVol}(X) \leq \frac{2}{\sqrt{3}} \frac{\mathrm{Vol}(X)}{l_0^2}$.

$$\mathrm{KVol}(X) := \mathrm{Vol}(X) \cdot \sup_{\alpha,\beta \text{ closed curves}} \frac{\mathrm{Int}(\alpha,\beta)}{l(\alpha)l(\beta)}$$

$\rightarrow$ Since the algebraic intersection is homology-invariant, it suffices to consider curves which minimizes the length in their homology class in the definition of KVol, that is (simple) **closed geodesics**.

$$\mathrm{KVol}(X) := \mathrm{Vol}(X) \cdot \sup_{\alpha,\beta \text{ closed curves}} \frac{\mathrm{Int}(\alpha,\beta)}{l(\alpha)l(\beta)}$$

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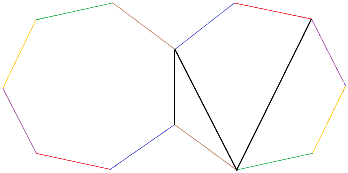
Definition

A *saddle connection* is a geodesic line starting and ending at singularities (not necessarily the same).



Saddle connections on the double heptagon.

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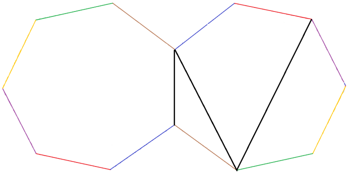
Since all the vertices are identified to the same point on the double heptagon, saddle connections are automatically closed.



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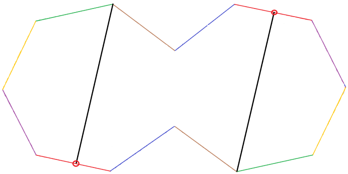

Another example of saddle connection.

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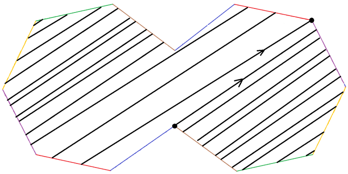

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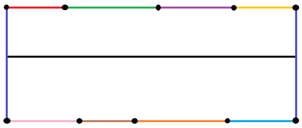
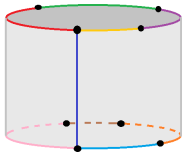
Proposition

Every closed curve is homologous to the union of finitely many saddle connections, and such unions of saddle connections minimizes the length in their homology class.





Every closed geodesic comes with a **cylinder** of freely homotopic geodesics. The boundary of the cylinder is a union of saddle connnections.

To compute KVol it suffices to consider (unions of) saddle connections.

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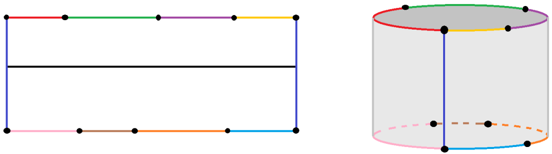

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To compute KVol it suffices to consider (unions of) saddle connections.

$\rightarrow$ For the rest of the talk, we will only consider surfaces with one singularity.

Question

How to compute the algebraic intersection of two saddle connections?

Proposition

Given $n \geq 5$ odd, the sides of the double regular n-gon are pairwise intersecting.

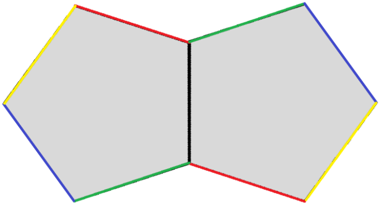

Figure: The double regular pentagon surface. All vertices are identified to the same point on the surface, which is the intersection point of the 5 systoles.

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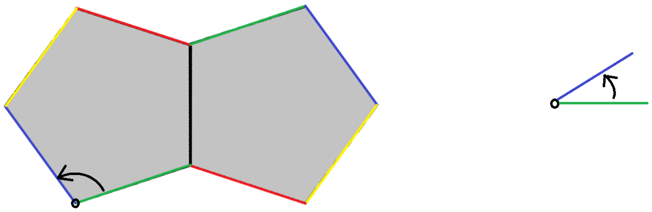

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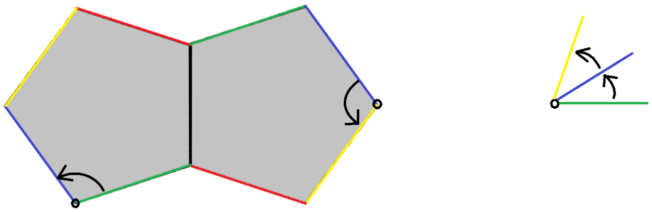

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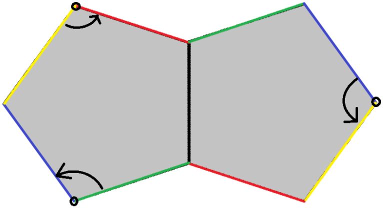

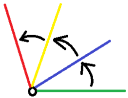

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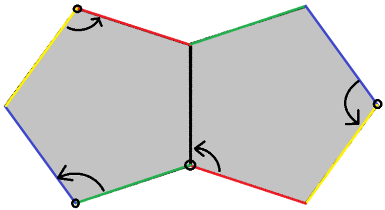

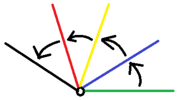

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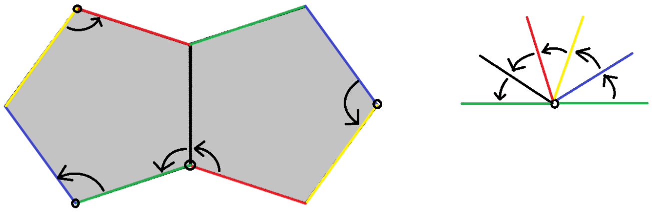

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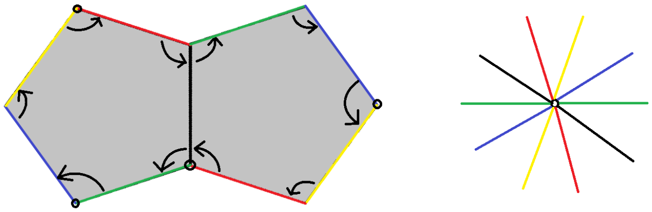

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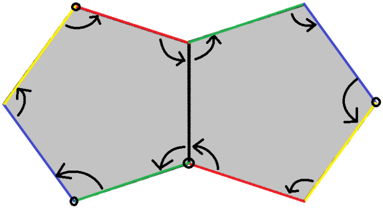

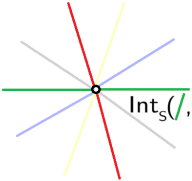

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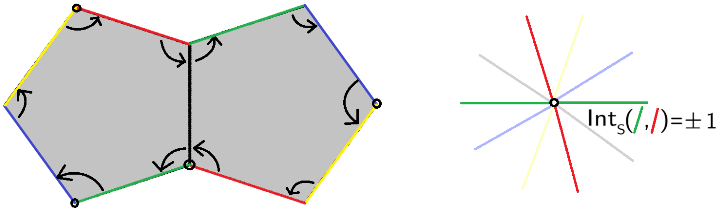

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Theorem (B., Lanneau, Massart, 2022)

For all $n \geq 5$ odd, the supremum in the definition of KVol on the double regular n-gon is achieved uniquely by pairs of (distinct) sides.

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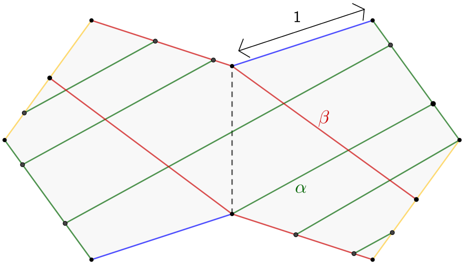
Idea of the proof:

Given two saddle connections α and β, we want to show:

$$\frac{\mathrm{Int}(\alpha, \beta)}{l(\alpha) l(\beta)} \leq 1$$

$\rightarrow$ We subdivide α and β into (non-closed) shorter segments for which we can control both the lengths and the intersections.



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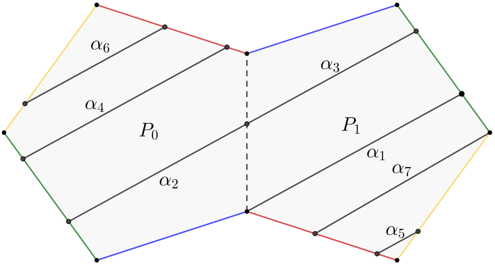
Idea of the proof:
Namely, we cut α (resp. β) each time it intersects a side of a polygon. We obtain a *polygonal decomposition*

$$\alpha = \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_k$$

$$\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_l$$



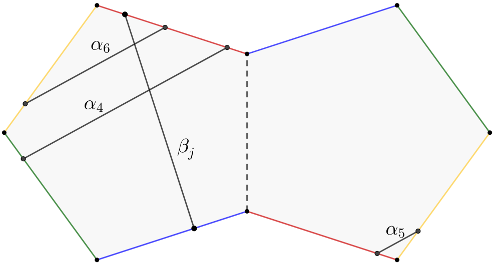
Theorem (B., Lanneau, Massart, 2022)

For all $n \geq 5$ odd, the supremum in the definition of KVol on the double regular n-gon is achieved uniquely by pairs of (distinct) sides.

We distinguish two types of segments: **adjacent** and **non-adjacent** segments.

Two non-adjacent segments α_i and β_j intersect at most once while their length is at least 1.

Although adjacent segments can be very short, we can glue them by pairs to control their length.



Although there are many technical difficulties, the proof extends to a more general setting:

Theorem (B.,Pasquinelli)

Let X be a translation surface constructed from a collection of polygons such that:

(H1) *All the polygons are **convex** with **obtuse angles**.*

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Then:

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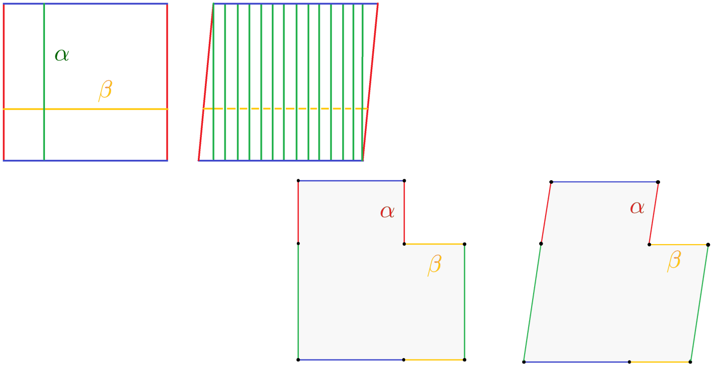
with equality if and only if there are two sides of length l_0 that represent closed curves and intersect once at a singularity. Here l_0 is the length of the shortest side of the polygons.

Theorem (B., Pasquinelli)

For coprime $m, n \geq 2$, KVol on $S_{m,n}$ is achieved by intersecting pairs of systoles.

Question

What happens to KVol when we deform a translation surface ?



Aim

Compute KVol on the $SL_2(\mathbb{R})$-orbit of Bouw-Möller surfaces.

KVol as a function on the $SL_2(\mathbb{R})$-orbit

We recall that for a given surface X:

$$KVol(X) = \underbrace{Vol(X)}_{=1} \cdot \sup_{\alpha,\beta} \frac{\mathrm{Int}(\alpha,\beta)}{l(\alpha)l(\beta)}$$

In a $SL_2(\mathbb{R})$-orbit, we rather want to consider closed curves in families. Namely, if $X = M \cdot S$, we can transport any curve α on S to a curve $\alpha(X)$ on X by the action of M.

This allows to see KVol on the $SL_2(\mathbb{R})$-orbit as a function which is a **supremum of functions**:

$$KVol(X) = \underbrace{Vol(X)}_{=1} \cdot \sup_{\alpha,\beta} \underbrace{\frac{\mathrm{Int}(\alpha(X),\beta(X))}{l(\alpha(X))l(\beta(X))}}_{f_{\alpha,\beta}(X)}$$

$$KVol(X) = \underbrace{Vol(X)}_{=1} \cdot \sup_{\alpha,\beta} \underbrace{\frac{\mathrm{Int}(\alpha(X), \beta(X))}{l_X(\alpha) l_X(\beta)}}_{f_{\alpha,\beta}(X)}$$

Lemma

If α and β are not parallel saddle connections, then

$$f_{\alpha,\beta}(X) = K(\alpha, \beta) \times \sin \mathrm{angle}(\alpha(X), \beta(X))$$

Where $K(\alpha, \beta) = \frac{\mathrm{Int}(\alpha,\beta)}{\alpha \wedge \beta}$ does not depend on X, and is invariant under the diagonal action of the Veech group.

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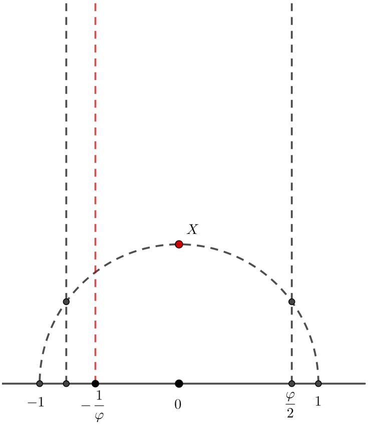
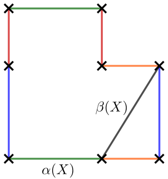
How to compute the angle between $\alpha(X)$ and $\beta(X)$?

Example

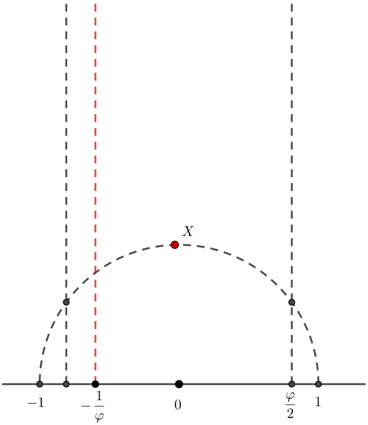


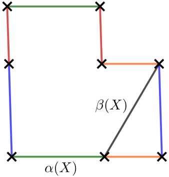


$t = 0$ $\quad M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$s = 0$ $\quad X = (0, 1)$

Example

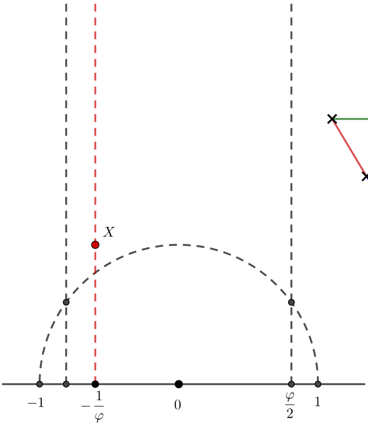


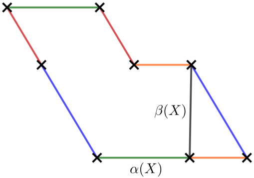


$t = 0 \qquad M = \begin{pmatrix} 1 & -0.03 \\ 0 & 1 \end{pmatrix}$

$s = -0.03 \quad X = (-0.03, 1)$

Example



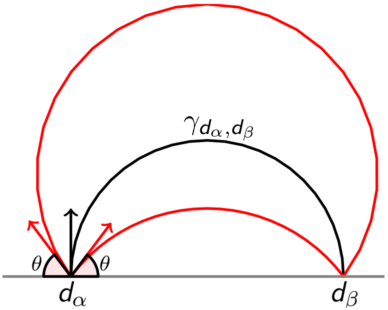


$t = 0$ $\quad M = \begin{pmatrix} 1 & -0.6 \\ 0 & 1 \end{pmatrix}$

$s = -0.6$ $\quad X = (-0.6, 1)$

Proposition (Cheboui-Kessi-Massart 2021)

Let d_α (resp. d_β) be the opposite of the co-slope of $\alpha(S)$ (resp. $\beta(S)$). Then the locus of surfaces X where the directions of $\alpha(X)$ and $\beta(X)$ make an angle θ is the ***banana neighborhood*** *of angle θ of the geodesic (d_α, d_β).*



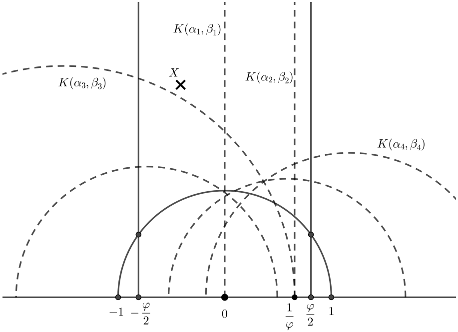
In black, the hyperbolic geodesic of endpoints d_α and d_β, which is the locus of surfaces where α and β are perpendicular. In red, the banana neighborhood of angle θ.

Remarks. 1. Here we work in the Teichmüller space.
2. The angles are not oriented

Geometric interpretation

If parallel saddle connections are non-intersecting, then KVol is a supremum of fonctions "*constant* $\times$ *sinus*".

$$KVol(X) = Vol(X) \cdot \sup_{\alpha,\beta}[K(\alpha,\beta)\sin\theta(X,\alpha,\beta)].$$



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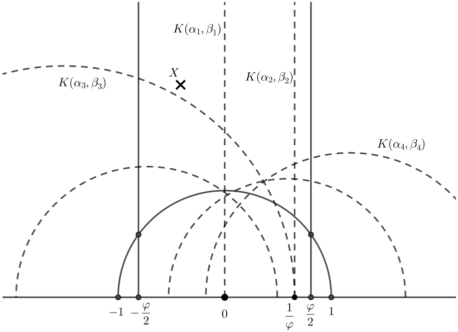
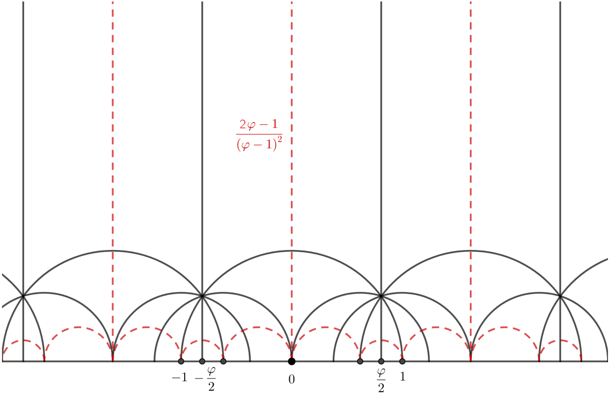
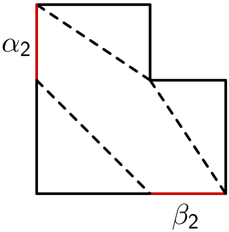


The pairs (α,β) that will achieve the supremum are most likely to be those for which $K(\alpha,\beta)$ is big.

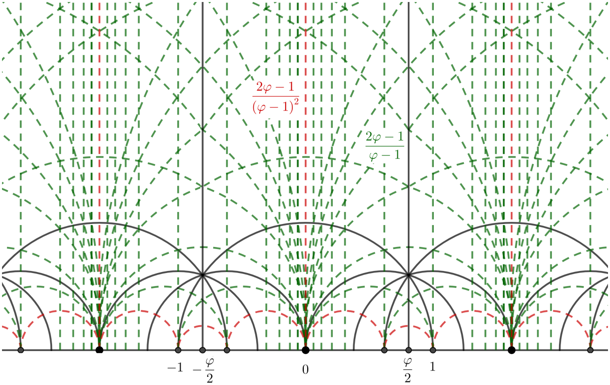
The double pentagon

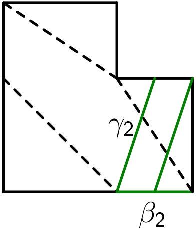




The pairs of curves (α, β) having maximal associated constant are (α_2, β_2) and their images by the diagonal action of the Veech group.

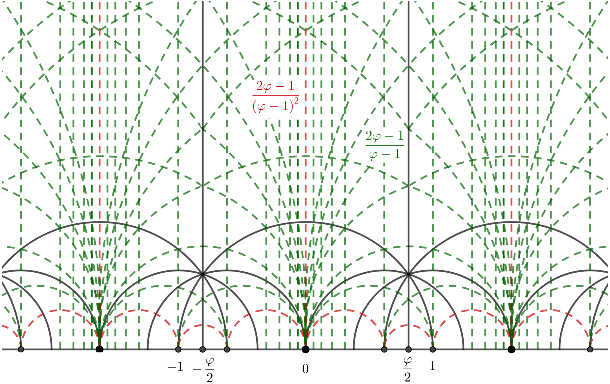
The double pentagon

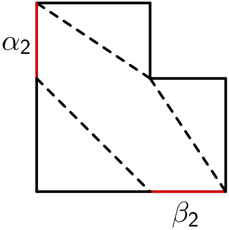




The pairs of curves (α, β) having second maximal associated constant are (β_2, γ_k), $k \in \mathbb{N}^\star$ and their images by the diagonal action of the Veech group.

The double pentagon

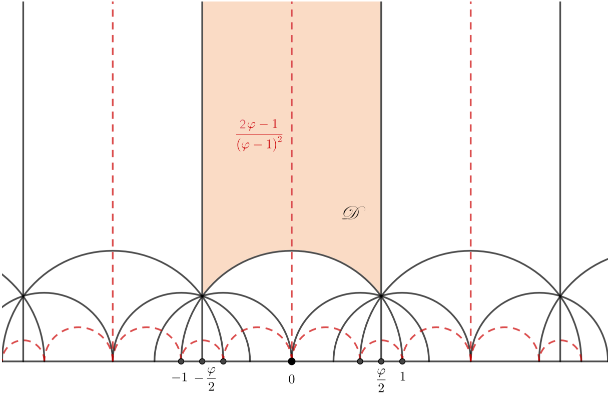


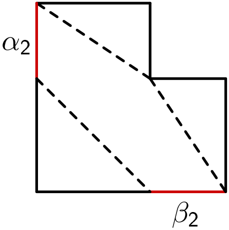


Theorem (B.,Lanneau,Massart, 2022)

To compute KVol, we only care about the red geodesics.

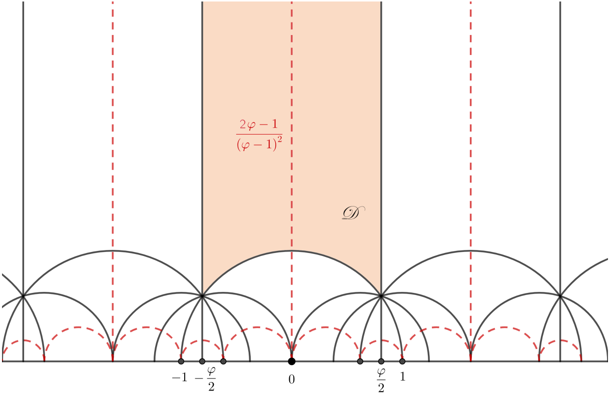
The double pentagon

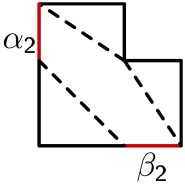


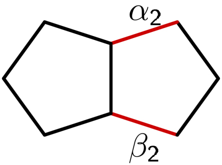


Theorem (B.,Lanneau,Massart, 2022)

$$\forall X \in \mathcal{D}, \forall \alpha, \beta, K(\alpha,\beta)\sin\theta(X,\alpha,\beta) \leq \underbrace{\frac{2\varphi-1}{(\varphi-1)^2}\sin\theta(X,\alpha_2,\beta_2)}_{=\mathrm{KVol}(X)}$$







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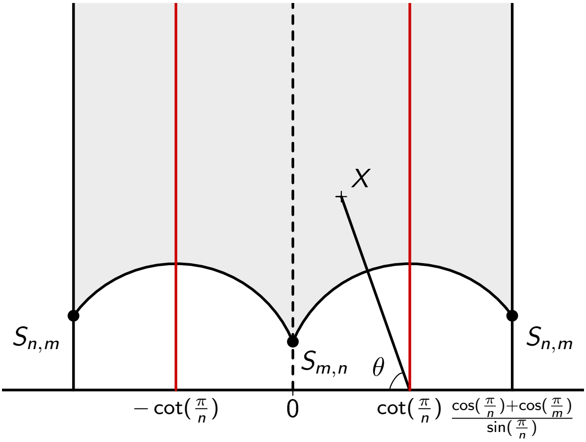
With I.Pasquinelli, we generalize this result to Bouw-Möller surfaces with a single singularity.



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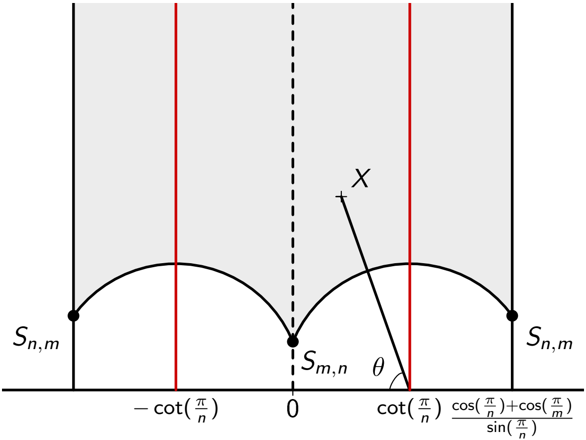

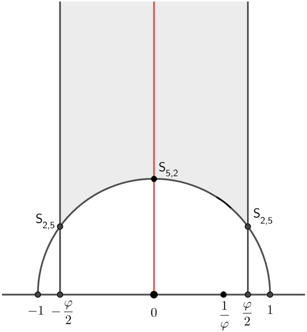

If $n = 2$, we have $\cot \frac{\pi}{n} = -\cot \frac{\pi}{n} = 0$:

$\longrightarrow$ the two red geodesics are merged.

Thanks for your attention